

COHOMOGENEITY ONE ACTIONS ON SYMMETRIC SPACES OF NONCOMPACT TYPE

JÜRGEN BERNDT AND HIROSHI TAMARU

ABSTRACT. An isometric action of a Lie group on a Riemannian manifold is of cohomogeneity one if the corresponding orbit space is one-dimensional. In this article we develop a conceptual approach to the classification of cohomogeneity one actions on Riemannian symmetric spaces of noncompact type in terms of orbit equivalence. As a consequence, we find many new examples of cohomogeneity one actions on Riemannian symmetric spaces of noncompact type. We apply our conceptual approach to derive explicit classifications of cohomogeneity one actions on some symmetric spaces.

1. INTRODUCTION

The cohomogeneity of an isometric action on a Riemannian manifold is the rank of the normal bundle of a principal orbit of the action. Thus, for a cohomogeneity one action, the principal orbits are hypersurfaces. Cohomogeneity one actions have been of much recent interest in the context of constructing geometric structures on manifolds.

The main focus in this article is on the classification of such actions on Riemannian symmetric spaces. A remarkable result by Hsiang and Lawson ([11]) states that every cohomogeneity one action on the round sphere S^n is orbit equivalent to the action on the sphere S^n which is induced from the isotropy representation of an $(n+1)$ -dimensional Riemannian symmetric space of rank two. Cohomogeneity one actions on the other compact simply connected Riemannian symmetric spaces of rank one, that is, the projective spaces over the normed real division algebras \mathbb{C} , \mathbb{H} and \mathbb{O} , were obtained by Takagi ([25]) and Iwata ([13],[14]). Kollross ([18]) derived the classification of cohomogeneity one actions on irreducible compact simply connected Riemannian symmetric spaces of higher rank. The classification for the reducible case is still outstanding.

In the noncompact case one needs to develop different techniques due to the noncompactness of the isometry groups. This can already be seen when considering cohomogeneity one actions on the Euclidean space \mathbb{E}^n . A group theoretical approach as in the compact case leads immediately to difficulties. However, there is a simple geometric solution to the problem. A principal orbit of a cohomogeneity one action is a hypersurface with constant principal curvatures, also known as an isoparametric hypersurface. Isoparametric hypersurfaces in Euclidean spaces were classified by Somigliana ([24]), Levi-Civita ([19]) and Segre ([23]), and it is easy to verify from their results that all complete isoparametric hypersurfaces in Euclidean spaces are homogeneous and hence principal orbits of cohomogeneity one actions. A similar approach leads to the classification of cohomogeneity

2010 *Mathematics Subject Classification.* Primary 53C35; Secondary 57S20.

Keywords. Riemannian symmetric spaces of noncompact type, cohomogeneity one actions, singular orbits, parabolic subgroups.

one actions on real hyperbolic spaces by using the classification of isoparametric hypersurfaces in real hyperbolic spaces by Cartan ([7]). However, this approach is successful only in these two cases. For example, the classification of hypersurfaces with constant principal curvatures in complex hyperbolic spaces is not yet known. In this article we present a conceptual approach for classifying cohomogeneity one actions on Riemannian symmetric spaces of noncompact type up to orbit equivalence. Two actions are said to be orbit equivalent if there exists an isometry of the space mapping the orbits of one action onto the orbits of the other action.

The orbit space of a cohomogeneity one action of a connected Lie group on a connected complete Riemannian manifold M is homeomorphic to the closed bounded interval $[0, 1]$, the closed unbounded interval $[0, \infty)$, the circle S^1 or the real line \mathbb{R} , each of them equipped with their standard topology. If M is a Riemannian symmetric space of noncompact type, then for topological reasons the orbit space must be homeomorphic to either \mathbb{R} or $[0, \infty)$. In the first case the orbits form a Riemannian foliation on M , and in the second case there is exactly one singular orbit and the principal orbits are the tubes around this singular orbit.

Let $M = G/K$ be a connected Riemannian symmetric space of noncompact type and $r = \text{rank}(M)$, where G is the identity component of the isometry group of M and K is the isotropy subgroup of G at a point $o \in M$. Let H be a connected subgroup of G which acts on M with cohomogeneity one. The case when the orbits of H form a Riemannian foliation on M has been dealt with by the authors in [3] for irreducible symmetric spaces M . We therefore assume that the action has a singular orbit W . Without loss of generality we may assume that $o \in W$. The subgroup H is contained in a connected maximal proper subgroup L of G . It follows from work by Mostow ([21]) that L is either reductive or the identity component of a parabolic subgroup of G . For the reductive case we show in Theorem 3.2 that H and L are orbit equivalent and that the singular orbit W is a totally geodesic submanifold in M . We now assume that L is the connected identity component of a parabolic subgroup of G .

The conjugacy classes of parabolic subgroups of G can be parametrized by the subsets Φ of a set $\Lambda = \{\alpha_1, \dots, \alpha_r\}$ of simple roots of a restricted root system of the semisimple Lie algebra \mathfrak{g} of G . The maximal proper parabolic subgroups correspond to subsets Φ of Λ with cardinality $|\Phi|$ equal to $r - 1$. For $\Phi = \emptyset$ we obtain a minimal parabolic subgroup of G . Let Q_Φ be the parabolic subgroup of G associated with the subset Φ of Λ . We construct new examples of cohomogeneity one actions on M from the Langlands decomposition and from the Chevalley decomposition of Q_Φ .

The Langlands decomposition is of the form $Q_\Phi = M_\Phi A_\Phi N_\Phi$, where M_Φ is reductive, A_Φ is abelian and N_Φ is nilpotent. The orbit $B_\Phi = M_\Phi \cdot o$ is a semisimple Riemannian symmetric space of noncompact type with $\text{rank}(B_\Phi) = |\Phi|$, unless $\Phi = \emptyset$ in which case the orbit consists just of the point o . The symmetric space B_Φ is embedded totally geodesically in M and is also known as a boundary component of M as it arises naturally in the maximal Satake compactification of M . The orbit $A_\Phi \cdot o$ is a Euclidean space $\mathbb{R}^{r-|\Phi|}$ of dimension $r - |\Phi|$ embedded in M as a totally geodesic submanifold. If H_Φ is a connected subgroup of the isometry group of B_Φ acting on B_Φ with cohomogeneity one, then $H = H_\Phi A_\Phi N_\Phi$ is a connected subgroup of $Q_\Phi \subset G$ acting on M with cohomogeneity one. We call this the canonical extension of the cohomogeneity one action on the boundary component B_Φ to the symmetric space M .

The Chevalley decomposition is of the form $Q_\Phi = L_\Phi N_\Phi$, where $L_\Phi = M_\Phi A_\Phi$ is reductive. The orbit $F_\Phi = L_\Phi \cdot o$ is isometric to the Riemannian product $B_\Phi \times \mathbb{E}^{r-|\Phi|}$ and embedded in M as a totally geodesic submanifold. Let \mathfrak{n}_Φ be the Lie algebra of N_Φ , and denote by H^Φ the sum of the dual root vectors of the simple roots in $\Lambda \setminus \Phi$. The vector H^Φ induces a gradation $\bigoplus_{\nu \geq 1} \mathfrak{n}_\Phi^\nu$ of \mathfrak{n}_Φ by defining \mathfrak{n}_Φ^ν as the sum of all root spaces corresponding to positive roots α with $\alpha(H^\Phi) = \nu \geq 1$. Let \mathfrak{v} be a subspace of \mathfrak{n}_Φ^1 with dimension ≥ 2 . Then $\mathfrak{n}_{\Phi, \mathfrak{v}} = \mathfrak{n}_\Phi \ominus \mathfrak{v}$ is a subalgebra of \mathfrak{n}_Φ . Denote by $N_{\Phi, \mathfrak{v}}$ the corresponding connected subgroup of N_Φ . Assume that the normalizer $N_{L_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ of $\mathfrak{n}_{\Phi, \mathfrak{v}}$ in L_Φ acts transitively on F_Φ and that the normalizer $N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ of $\mathfrak{n}_{\Phi, \mathfrak{v}}$ in $K_\Phi = L_\Phi \cap K$ acts transitively on the unit sphere in \mathfrak{v} . Note that $N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ coincides with the normalizer $N_{K_\Phi}(\mathfrak{v})$ of \mathfrak{v} in K_Φ . Then $H_{\Phi, \mathfrak{v}} = N_{L_\Phi}^\circ(\mathfrak{n}_{\Phi, \mathfrak{v}})N_{\Phi, \mathfrak{v}}$ is a connected subgroup of $Q_\Phi = L_\Phi N_\Phi$ which acts on M with cohomogeneity one and singular orbit $H_{\Phi, \mathfrak{v}} \cdot o$. We provide some explicit examples of such actions below.

We put $\mathfrak{a} = \mathfrak{a}_0$ and $\mathfrak{n} = \mathfrak{n}_0$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of \mathfrak{g} , and the connected solvable subgroup AN of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ acts simply transitively on M . Therefore M is isometric to AN equipped with a suitable left-invariant Riemannian metric.

Let ℓ be a one-dimensional linear subspace of \mathfrak{a} . Then $\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ is a codimension one subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and hence the connected subgroup H_ℓ of G with Lie algebra \mathfrak{h}_ℓ acts on M with cohomogeneity one. The orbits form a Riemannian foliation on M whose orbits are pairwise isometrically congruent.

Let ℓ be a one-dimensional linear subspace of a simple root space \mathfrak{g}_{α_i} . Then $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ is a codimension one subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, and hence the connected subgroup of G with Lie algebra $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ acts on M with cohomogeneity one. The orbits of this action form a Riemannian foliation on M , and there is exactly one minimal orbit. Moreover, assume that ℓ and ℓ' are two one-dimensional linear subspaces of \mathfrak{g}_{α_i} . Then the cohomogeneity one actions induced from $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ and $\mathfrak{a} \oplus (\mathfrak{n} \ominus \ell')$ are orbit equivalent. Therefore, for each choice of simple root $\alpha_i \in \Lambda$ we get exactly one cohomogeneity one action up to orbit equivalence. We denote by H_i one of the connected subgroups of G constructed in this manner.

We can now formulate the main result of this article.

Theorem 1.1. *Let $M = G/K$ be a connected irreducible Riemannian symmetric space of noncompact type and with rank r , and let H be a connected subgroup of G which acts on M with cohomogeneity one. Then either statement (1) or statement (2) holds:*

- (1) *The orbits form a Riemannian foliation on M and one of the following two cases holds:*
 - (i) *All orbits are isometrically congruent to each other, and the action of H is orbit equivalent to the action of H_ℓ for some one-dimensional linear subspace ℓ of \mathfrak{a} .*
 - (ii) *There exists exactly one minimal orbit, and the action of H is orbit equivalent to the action of H_i for some $i \in \{1, \dots, r\}$.*
- (2) *There exists exactly one singular orbit and one of the following two cases holds:*
 - (i) *H is contained in a maximal proper reductive subgroup L of G , the actions of H and L are orbit equivalent, and the singular orbit is totally geodesic in M .*
 - (ii) *H is contained in a maximal proper parabolic subgroup Q_Φ of G and one of the following two cases holds:*

- (a) *The action of H is orbit equivalent to the canonical extension of a cohomogeneity one action with a singular orbit on the boundary component B_Φ of M .*
- (b) *The action of H is orbit equivalent to a cohomogeneity one action on M given by $H_{\Phi, \mathfrak{v}}$ for some subspace $\mathfrak{v} \subset \mathfrak{n}_\Phi^\perp$ with $\dim \mathfrak{v} \geq 2$.*

Remarks. 1. Consider the Dynkin diagram associated to the simple roots Λ . Each symmetry σ of the Dynkin diagram gives rise to an automorphism F_σ of \mathfrak{a} .

In case (1)(i), assume that ℓ and ℓ' are two one-dimensional linear subspaces of \mathfrak{a} . Then the cohomogeneity one actions induced from $(\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ and $(\mathfrak{a} \ominus \ell') \oplus \mathfrak{n}$ are orbit equivalent if and only if there exists a Dynkin diagram symmetry σ such that $F_\sigma(\ell) = \ell'$. The cohomogeneity one actions of type (1)(i) are therefore parametrized by $\mathbb{R}P^{r-1}/\mathfrak{S}$, where $\mathbb{R}P^{r-1}$ is the real projective space of the real vector space \mathfrak{a} and \mathfrak{S} is the finite group of automorphisms of $\mathbb{R}P^{r-1}$ which is induced by the automorphisms F_σ of \mathfrak{a} . For details we refer to Theorem 3.5 in [3].

In case (1)(ii), let $i, j \in \{1, \dots, r\}$. The actions of H_i and H_j are orbit equivalent if and only if there exists a Dynkin diagram symmetry σ such that $\sigma(\alpha_i) = \alpha_j$. The cohomogeneity one actions of type (1)(ii) are therefore parametrized by $\{1, \dots, r\}/\mathfrak{S}$, where \mathfrak{S} is the finite group of automorphisms of the Dynkin diagram. For details we refer to Theorem 4.8 in [3].

2. There is a well-known concept of duality between Riemannian symmetric spaces of noncompact type and Riemannian symmetric space of compact type. A totally geodesic submanifold W of M corresponds via this duality to a totally geodesic submanifold W^* in the dual Riemannian symmetric space M^* of compact type. A cohomogeneity one action of H on M with a totally geodesic singular orbit W then gives rise to a cohomogeneity one action on M^* of some connected subgroup H^* of the isometry group of M^* . Using the classification by Kollross ([18]) of cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type, and the concept of reflective submanifolds, the authors determined in [4] all totally geodesic submanifolds in irreducible Riemannian symmetric spaces of noncompact type which arise as a singular orbit of a cohomogeneity one action. There are exactly five totally geodesic submanifolds which are not reflective, and mysteriously these are all related to the exceptional Lie group G_2 . We refer to [4] for further details.

We point out here that the explicit classification of totally geodesic submanifolds in reducible Riemannian symmetric spaces of noncompact type which arise as a singular orbit of a cohomogeneity one action is still an open problem.

3. The concept of canonical extension in (2)(ii)(a) suggests of course a rank reduction for the classification. However, since the boundary component B_Φ can be reducible, we encounter the same problem we discussed at the end of the previous remark.

4. We do not have an explicit classification of the groups $H_{\Phi, \mathfrak{v}}$ arising in (2)(ii)(b). However, our calculations indicate that there are only few examples which cannot be constructed via (2)(i) or (2)(ii)(a). The first author and Brück constructed in [1] new examples on the hyperbolic spaces over the normed real division algebras \mathbb{C} , \mathbb{H} and \mathbb{O} . The authors proved in [5] that there are no further examples in the cases of \mathbb{C} and \mathbb{O} , but for \mathbb{H} the problem remains open. In this article we construct two new cohomogeneity one actions with this method, one on G_2^2/SO_4 and one on $G_2^\mathbb{C}/G_2$. Although we checked many other symmetric spaces, we could not find any further examples and start to believe that

there are none apart from the obvious ones on reducible symmetric spaces obtained from the known examples on irreducible symmetric spaces.

This article is organised as follows. In Section 2 we outline basic material about parabolic subalgebras of semisimple real Lie algebras, and relate this to the geometry of Riemannian symmetric spaces of noncompact type. In Section 3 we show first that a proper maximal reductive subgroup of the isometry group of a Riemannian symmetric space of noncompact type cannot act transitively on the space. We then relate cohomogeneity one actions to actions of reductive and parabolic subgroups. In Section 4 we present two new methods for constructing cohomogeneity one actions with a singular orbit on Riemannian symmetric spaces of noncompact type. In Section 5 we proof the main result of this article. In Section 6 we apply the main result to derive explicit classifications of cohomogeneity one actions on some Riemannian symmetric spaces of noncompact type and rank 2.

2. PARABOLIC SUBALGEBRAS

In this section we recall the construction of the parabolic subalgebras of real semisimple Lie algebras (see e.g. [6], [17] and [22] for more details and proofs) and discuss some aspects of their geometry.

Let \mathfrak{g} be a real semisimple Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} . Let θ be the corresponding Cartan involution on \mathfrak{g} and B the Cartan-Killing form on \mathfrak{g} . Then $\langle X, Y \rangle = -B(X, \theta Y)$ is a positive definite inner product on \mathfrak{g} . If V, W are linear subspaces of \mathfrak{g} and $V \subset W$, we denote by $W \ominus V$ the orthogonal complement of V in W with respect to the inner product, that is, $W \ominus V = \{w \in W \mid \langle w, v \rangle = 0 \text{ for all } v \in V\}$.

Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{p} and denote by \mathfrak{a}^* the dual space of \mathfrak{a} . For each $\alpha \in \mathfrak{a}^*$ we define $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$. If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is a restricted root and \mathfrak{g}_α a restricted root space of \mathfrak{g} with respect to \mathfrak{a} . We denote by Σ the set of restricted roots with respect to \mathfrak{a} . The subspace \mathfrak{g}_0 coincides with $\mathfrak{k}_0 \oplus \mathfrak{a}$, where \mathfrak{k}_0 is the centralizer of \mathfrak{a} in \mathfrak{k} . We recall that $\mathfrak{k}_0 = \{0\}$ if and only if \mathfrak{g} is a split real form of its complexification $\mathfrak{g}^{\mathbb{C}}$. The direct sum decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

is the restricted root space decomposition of \mathfrak{g} with respect to \mathfrak{a} . For each $\alpha \in \Sigma$ we define the root vector $H_\alpha \in \mathfrak{a}$ corresponding to α by the equation $\alpha(H) = \langle H_\alpha, H \rangle$ for all $H \in \mathfrak{a}$.

Let $\{\alpha_1, \dots, \alpha_r\} = \Lambda \subset \Sigma$ be a set of simple roots of Σ , and denote by Σ^+ the corresponding set of all positive roots in Σ . The subalgebra

$$\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

is nilpotent and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of \mathfrak{g} .

We will now associate to each subset Φ of Λ a parabolic subalgebra \mathfrak{q}_Φ of \mathfrak{g} . Let Φ be a subset of Λ . We denote by Σ_Φ the root subsystem of Σ generated by Φ , that is, Σ_Φ is the intersection of Σ and the linear span of Φ , and put $\Sigma_\Phi^+ = \Sigma_\Phi \cap \Sigma^+$. We define a reductive

subalgebra \mathfrak{l}_Φ of \mathfrak{g} and a nilpotent subalgebra \mathfrak{n}_Φ of \mathfrak{g} by

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha \right) \quad \text{and} \quad \mathfrak{n}_\Phi = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+} \mathfrak{g}_\alpha.$$

Let

$$\mathfrak{a}_\Phi = \bigcap_{\alpha \in \Phi} \ker \alpha$$

be the split component of \mathfrak{l}_Φ and define $\mathfrak{a}^\Phi = \mathfrak{a} \ominus \mathfrak{a}_\Phi$. Then \mathfrak{a}_Φ is an abelian subalgebra of \mathfrak{g} and \mathfrak{l}_Φ is the centralizer and the normalizer of \mathfrak{a}_Φ in \mathfrak{g} . Since $[\mathfrak{l}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi$,

$$\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$$

is a subalgebra of \mathfrak{g} , the so-called parabolic subalgebra of \mathfrak{g} associated with the subsystem Φ of Λ . The decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ is the Chevalley decomposition of the parabolic subalgebra \mathfrak{q}_Φ .

We now define a reductive subalgebra \mathfrak{m}_Φ of \mathfrak{g} by $\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi$. The subalgebra \mathfrak{m}_Φ normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, and $\mathfrak{g}_\Phi = [\mathfrak{m}_\Phi, \mathfrak{m}_\Phi] = [\mathfrak{l}_\Phi, \mathfrak{l}_\Phi]$ is a semisimple subalgebra of \mathfrak{g} . The center \mathfrak{z}_Φ of \mathfrak{m}_Φ is contained in \mathfrak{k}_0 and induces the direct sum decomposition $\mathfrak{m}_\Phi = \mathfrak{z}_\Phi \oplus \mathfrak{g}_\Phi$. The decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

is the Langlands decomposition of the parabolic subalgebra \mathfrak{q}_Φ .

For $\Phi = \emptyset$ we have $\mathfrak{l}_\emptyset = \mathfrak{g}_0$, $\mathfrak{m}_\emptyset = \mathfrak{k}_0$, $\mathfrak{a}_\emptyset = \mathfrak{a}$ and $\mathfrak{n}_\emptyset = \mathfrak{n}$. In this case $\mathfrak{q}_\emptyset = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n} = \mathfrak{g}_0 \oplus \mathfrak{n}$ is a minimal parabolic subalgebra of \mathfrak{g} . For $\Phi = \Lambda$ we obtain $\mathfrak{l}_\Lambda = \mathfrak{m}_\Lambda = \mathfrak{g}$ and $\mathfrak{a}_\Lambda = \mathfrak{n}_\Lambda = \{0\}$. The proper maximal parabolic subalgebras of \mathfrak{g} are precisely those parabolic subalgebras for which the cardinality $|\Lambda \setminus \Phi|$ of $\Lambda \setminus \Phi$ is equal to one. The proper maximal parabolic subalgebras can therefore be parametrized by the simple roots in Λ .

Each parabolic subalgebra of \mathfrak{g} is conjugate in \mathfrak{g} to \mathfrak{q}_Φ for some subset Φ of Λ . The set of conjugacy classes of parabolic subalgebras of \mathfrak{g} therefore has 2^r elements, where $r = |\Lambda|$ is the real rank of \mathfrak{g} . Two parabolic subalgebras \mathfrak{q}_{Φ_1} and \mathfrak{q}_{Φ_2} of \mathfrak{g} are conjugate in the full automorphism group $\text{Aut}(\mathfrak{g})$ of \mathfrak{g} if and only if there exists an automorphism F of the Dynkin diagram associated to Λ with $F(\Phi_1) = \Phi_2$.

For each $\alpha \in \Sigma$ we define $\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha)$ and $\mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha)$. Then we have $\mathfrak{k}_{-\alpha} = \mathfrak{k}_\alpha$, $\mathfrak{p}_{-\alpha} = \mathfrak{p}_\alpha$ and $\mathfrak{k}_\alpha \oplus \mathfrak{p}_\alpha = \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$ for all $\alpha \in \Sigma$. It is easy to see that the subspaces

$$\mathfrak{p}_\Phi = \mathfrak{l}_\Phi \cap \mathfrak{p} = \mathfrak{a} \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{p}_\alpha \right) \quad \text{and} \quad \mathfrak{b}_\Phi = \mathfrak{m}_\Phi \cap \mathfrak{p} = \mathfrak{g}_\Phi \cap \mathfrak{p} = \mathfrak{a}^\Phi \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{p}_\alpha \right)$$

are Lie triple systems in \mathfrak{p} . We define a subalgebra \mathfrak{k}_Φ of \mathfrak{k} by

$$\mathfrak{k}_\Phi = \mathfrak{q}_\Phi \cap \mathfrak{k} = \mathfrak{l}_\Phi \cap \mathfrak{k} = \mathfrak{m}_\Phi \cap \mathfrak{k} = \mathfrak{k}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{k}_\alpha \right).$$

Then we have

$$[\mathfrak{k}_\Phi, \mathfrak{m}_\Phi] \subset \mathfrak{m}_\Phi, \quad [\mathfrak{k}_\Phi, \mathfrak{a}_\Phi] = \{0\}, \quad [\mathfrak{k}_\Phi, \mathfrak{n}_\Phi] \subset \mathfrak{n}_\Phi.$$

These three relations will be important for our understanding of cohomogeneity one actions on M . Moreover, $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi) \oplus \mathfrak{b}_\Phi$ is a Cartan decomposition of the semisimple subalgebra \mathfrak{g}_Φ of \mathfrak{g} and \mathfrak{a}^Φ is a maximal abelian subspace of \mathfrak{b}_Φ . If we define $(\mathfrak{g}_\Phi)_0 =$

$(\mathfrak{g}_\Phi \cap \mathfrak{k}_0) \oplus \mathfrak{a}^\Phi$, then $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi)_0 \oplus (\bigoplus_{\alpha \in \Sigma_\Phi} \mathfrak{g}_\alpha)$ is the restricted root space decomposition of \mathfrak{g}_Φ with respect to \mathfrak{a}^Φ and Φ is the corresponding set of simple roots. Since $\mathfrak{m}_\Phi = \mathfrak{z}_\Phi \oplus \mathfrak{g}_\Phi$ and $\mathfrak{z}_\Phi \subset \mathfrak{k}_0$, we see that $\mathfrak{g}_\Phi \cap \mathfrak{k}_0 = \mathfrak{k}_0 \ominus \mathfrak{z}_\Phi$.

We now relate these algebraic constructions to the geometry of symmetric spaces of noncompact type. Let $M = G/K$ be the connected Riemannian symmetric space of noncompact type associated with the pair $(\mathfrak{g}, \mathfrak{k})$. The Riemannian metric on M is the one which is induced from the $\text{Ad}(K)$ -invariant inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{p} . Then $G = I^o(M)$ is the connected component of the isometry group of M containing the identity and K is a maximal compact subgroup of G . The Lie algebra of G and K coincides with \mathfrak{g} and \mathfrak{k} , respectively. We denote by $o \in M$ the unique fixed point of K , that is, o is the point in M for which the stabilizer of G at o coincides with K . We identify the subspace \mathfrak{p} in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with the tangent space $T_o M$ of M at o in the usual way. The rank of the symmetric space M coincides with $r = |\Lambda|$.

Let $\text{Exp} : \mathfrak{g} \rightarrow G$ be the Lie exponential map of \mathfrak{g} . Then $A = \text{Exp}(\mathfrak{a})$ and $N = \text{Exp}(\mathfrak{n})$ is a simply connected closed subgroup of G with Lie algebra \mathfrak{a} and \mathfrak{n} , respectively, A is abelian and N is nilpotent. The orbit $A \cdot o$ is an r -dimensional Euclidean space \mathbb{E}^r embedded totally geodesically into M , and the orbit $N \cdot o$ is a horocycle in M . The Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of \mathfrak{g} induces an Iwasawa decomposition $G = KAN$ of G . The solvable Lie group AN acts simply transitively on the symmetric space M .

Let Φ be a subset of Λ and $r_\Phi = |\Phi|$. We denote by A_Φ the connected abelian subgroup of G with Lie algebra \mathfrak{a}_Φ and by N_Φ the connected nilpotent subgroup of G with Lie algebra \mathfrak{n}_Φ . The centralizer $L_\Phi = Z_G(\mathfrak{a}_\Phi)$ of \mathfrak{a}_Φ in G is a reductive subgroup of G with Lie algebra \mathfrak{l}_Φ . Moreover, L_Φ normalizes N_Φ , and hence $Q_\Phi = L_\Phi N_\Phi$ is a subgroup of G with Lie algebra \mathfrak{q}_Φ . The subgroup Q_Φ coincides with the normalizer $N_G(\mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi)$ of $\mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$ in G , and hence Q_Φ is a closed subgroup of G . The subgroup Q_Φ is the parabolic subgroup of G associated with the subsystem Φ of Λ . We denote by Q_Φ^o the connected component of Q_Φ containing the identity transformation.

Let G_Φ be the connected subgroup of G with Lie algebra \mathfrak{g}_Φ . Since \mathfrak{g}_Φ is semisimple, G_Φ is a semisimple subgroup of G . The intersection $K_\Phi = L_\Phi \cap K$ is a maximal compact subgroup of L_Φ and \mathfrak{k}_Φ is the Lie algebra of K_Φ . The adjoint group $\text{Ad}(L_\Phi)$ normalizes \mathfrak{g}_Φ , and consequently $M_\Phi = K_\Phi G_\Phi$ is a subgroup of L_Φ . One can show that M_Φ is a closed reductive subgroup of L_Φ , K_Φ is a maximal compact subgroup of M_Φ , and the center Z_Φ of M_Φ is a compact subgroup of K_Φ . The Lie algebra of M_Φ is \mathfrak{m}_Φ and L_Φ is isomorphic to the Lie group direct product $M_\Phi \times A_\Phi$. The multiplication $M_\Phi \times A_\Phi \times N_\Phi \rightarrow Q_\Phi$ is an analytic diffeomorphism, and the group structure is given by

$$(m, a, n)(m', a', n') = (mm', aa', (m'a')^{-1}n(m'a')n').$$

The parabolic subgroup Q_Φ acts transitively on M and the isotropy subgroup at o is K_Φ , that is, $M = Q_\Phi/K_\Phi$.

Since $\mathfrak{g}_\Phi = (\mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi) \oplus \mathfrak{b}_\Phi$ is a Cartan decomposition of the semisimple subalgebra \mathfrak{g}_Φ , we have $[\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] = \mathfrak{g}_\Phi \cap \mathfrak{k}_\Phi$. Thus G_Φ is the connected closed subgroup of G with Lie algebra $[\mathfrak{b}_\Phi, \mathfrak{b}_\Phi] \oplus \mathfrak{b}_\Phi$. Since \mathfrak{b}_Φ is a Lie triple system in \mathfrak{p} , the orbit $B_\Phi = G_\Phi \cdot o$ of the G_Φ -action on M containing o is a connected totally geodesic submanifold of M with $T_o B_\Phi = \mathfrak{b}_\Phi$. If $\Phi = \emptyset$, then $B_\Phi = \{o\}$, otherwise B_Φ is a Riemannian symmetric space of noncompact type and $\text{rank}(B_\Phi) = r_\Phi$, and

$$B_\Phi = G_\Phi \cdot o = G_\Phi / (G_\Phi \cap K_\Phi) = M_\Phi \cdot o = M_\Phi / K_\Phi.$$

The submanifold B_Φ is also known as a boundary component of M in the context of the maximal Satake compactification of M (see e.g. [6]).

Clearly, \mathfrak{a}_Φ is a Lie triple system as well, and the corresponding totally geodesic submanifold is a Euclidean space

$$\mathbb{E}^{r-r_\Phi} = A_\Phi \cdot o.$$

Finally, $\mathfrak{p}_\Phi = \mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi$ is a Lie triple system, and the corresponding totally geodesic submanifold F_Φ is the symmetric space

$$F_\Phi = L_\Phi \cdot o = L_\Phi / K_\Phi = (M_\Phi \times A_\Phi) / K_\Phi = B_\Phi \times \mathbb{E}^{r-r_\Phi}.$$

The analytic diffeomorphism $M_\Phi \times A_\Phi \times N_\Phi \rightarrow Q_\Phi$ induces an analytic diffeomorphism

$$B_\Phi \times A_\Phi \times N_\Phi \rightarrow M, (m \cdot o, a, n) \mapsto (man) \cdot o,$$

known as a horospherical decomposition of the symmetric space M . The action of Q_Φ on M is given by

$$Q_\Phi \times M \rightarrow M, ((m, a, n), (m' \cdot o, a', n')) \mapsto ((mm') \cdot o, aa', (m'a')^{-1}n(m'a')n').$$

3. MAXIMAL REDUCTIVE AND PARABOLIC SUBGROUPS

In this section we relate cohomogeneity one actions on M to actions of reductive and parabolic subgroups of G .

Proposition 3.1. *Every connected proper maximal reductive subgroup L of G has a totally geodesic orbit W in M with $\dim W < \dim M$. In particular, L cannot act transitively on M .*

Proof. Let \mathfrak{l} be the Lie algebra of L . As \mathfrak{g} is algebraic (see e.g. [22], p. 29, Corollary 4) and \mathfrak{l} is maximal in \mathfrak{g} , \mathfrak{l} is an algebraic subalgebra of \mathfrak{g} . Since \mathfrak{l} is a reductive algebraic subalgebra of \mathfrak{g} , there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of \mathfrak{g} such that $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{k}) \oplus (\mathfrak{l} \cap \mathfrak{p})$ (see e.g. [22], p. 207, Theorem 3.6). Let K be the maximal compact subgroup of G with Lie algebra \mathfrak{k} , and let $o \in M$ be the fixed point of K . Then the orbit $W = L \cdot o$ of L through o is a totally geodesic submanifold of M (see e.g. [2], Proposition 9.1.2). Assume that $\dim W = \dim M$, which means that $\mathfrak{p} = \mathfrak{l} \cap \mathfrak{p} \subset \mathfrak{l}$. Since \mathfrak{g} is semisimple and contains no nonzero compact ideals, we have $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ (see e.g. [22], p. 145, Proposition 3.5). This implies $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}] \subset [\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{l}$. Altogether this gives $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \subset \mathfrak{l}$ and hence $\mathfrak{l} = \mathfrak{g}$. As G is connected, this contradicts the assumption that L is a proper subgroup of G , and we conclude $\dim W < \dim M$. q.e.d.

Remarks. 1. It was shown by Karpelevic [16] that every connected semisimple subgroup of G has a totally geodesic orbit in M . This follows also from Theorem 6 proved by Mostow in [20]. A geometric proof for the semisimple case was recently given by Di Scala and Olmos in [8].

2. The corresponding statement for Riemannian symmetric spaces of compact type is not true. Consider for example the 4-dimensional sphere $S^4 = SO(5)/SO(4)$ as a subset of the 5-dimensional real vector space of all symmetric (3×3) -matrices with real coefficients and trace zero. By considering the action of $SO(3)$ on such matrices by conjugation one gets a cohomogeneity one action on S^4 with no totally geodesic orbit. The two singular orbits of this action are congruent to the Veronese embedding of the real projective plane \mathbb{RP}^2 into S^4 . In the compact case there also exist connected proper reductive subgroups

which act transitively. For example, $SU(n)$ ($n \geq 2$) is a connected proper reductive subgroup of $SO(2n)$ which acts transitively on $SO(2n)/SO(2n-1) = S^{2n-1}$.

Theorem 3.2. *Let M be a connected Riemannian symmetric space of noncompact type and H a connected subgroup of $G = I^\circ(M)$ acting on M with cohomogeneity one. Let L be a connected proper maximal subgroup of G with $H \subset L$. Then one of the following statements holds:*

- (1) *L is a reductive subgroup of G , the actions of H and L are orbit equivalent, and the action of H on M has a totally geodesic orbit W . Moreover, if M is irreducible and $M \neq \mathbb{R}H^n = SO_{1,n}^\circ/SO_n$, then W is a singular orbit.*
- (2) *L is the identity component of a parabolic subgroup of G .*

Proof. We denote by \mathfrak{l} the Lie algebra of L , by \mathfrak{r} the radical of \mathfrak{l} , and by \mathfrak{n} the nilradical of \mathfrak{l} . It is a well-known consequence of Lie's Theorem on solvable Lie algebras that $[\mathfrak{l}, \mathfrak{r}] \subset \mathfrak{n}$ (see e.g. [17], Corollary 1.41). Mostow has shown (see proof of Theorem 3.1 in [21]) that the nilradical \mathfrak{n} is trivial if and only if L is unimodular.

Let us first assume that L is unimodular. Then $[\mathfrak{l}, \mathfrak{r}] = 0$, which implies that \mathfrak{r} is contained in the center \mathfrak{z} of \mathfrak{l} . As the center of a Lie algebra is always contained in the radical of the Lie algebra, we conclude that the radical \mathfrak{r} of \mathfrak{l} coincides with the center \mathfrak{z} of \mathfrak{l} . Therefore \mathfrak{l} is a reductive Lie algebra. As $H \subset L$, the orbits of the action of H are contained in the orbits of the action of L . However, L cannot act transitively on M (see Proposition 3.1) and hence must act on M with cohomogeneity one. Since both L and H are connected, the orbits of H and L must therefore coincide, and Proposition 3.1 implies that H has a totally geodesic orbit W with $\dim W < \dim M$.

The real hyperbolic spaces $\mathbb{R}H^n$, $n \geq 2$, are the only irreducible Riemannian symmetric spaces of noncompact type which have a totally geodesic hypersurface (see e.g. [12]). Therefore, if M is irreducible and $M \neq \mathbb{R}H^n = SO_{1,n}^\circ/SO_n$, the totally geodesic orbit W must be a singular orbit of the action.

If L is not unimodular, then \mathfrak{l} is a parabolic subalgebra of \mathfrak{g} by a result of Mostow ([21]), and hence L is the identity component of a parabolic subgroup of G . Therefore L is conjugate to Q_Φ° for some subset Φ of Λ , and since L is a maximal proper subgroup of G , we have $|\Phi| = r - 1$. q.e.d.

Remark. The maximal reductive nonsemisimple subalgebras of real semisimple Lie algebras have been classified by Tao [27].

In view of Theorem 3.2 we now consider more thoroughly the case when \mathfrak{l} is a parabolic subalgebra of \mathfrak{g} .

4. THE PARABOLIC CASE

In this section we assume that \mathfrak{h} is contained in a parabolic subalgebra \mathfrak{l} of \mathfrak{g} . From Section 2 we know that \mathfrak{l} is conjugate to \mathfrak{q}_Φ for some subset Φ of Λ . Without loss of generality we assume that $\mathfrak{l} = \mathfrak{q}_\Phi$. Now consider the Langlands decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

of \mathfrak{q}_Φ and the corresponding horospherical decomposition

$$M \cong B_\Phi \times \mathbb{E}^{r-r_\Phi} \times N_\Phi \cong B_\Phi \times A_\Phi \times N_\Phi$$

of M . Note that for the second congruence we identify the Euclidean space \mathbb{E}^{r-r_Φ} and the abelian Lie group A_Φ via the simple transitive action of A_Φ on \mathbb{E}^{r-r_Φ} . We now construct two types of cohomogeneity one actions from the Langlands or horospherical decomposition.

4.1. Canonical extensions from boundary components. Let H_Φ be a connected subgroup of $I(B_\Phi)$ and denote by \mathfrak{h}_Φ the Lie algebra of H_Φ . Since $\mathfrak{h}_\Phi \subset \mathfrak{g}_\Phi \subset \mathfrak{m}_\Phi$ and \mathfrak{m}_Φ normalizes $\mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$, we see that

$$\mathfrak{h}_\Phi^\Lambda = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$$

is a subalgebra of $\mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = \mathfrak{q}_\Phi \subset \mathfrak{g}$. We call the connected subgroup H_Φ^Λ of the parabolic subgroup Q_Φ of G with Lie algebra $\mathfrak{h}_\Phi^\Lambda$ the *canonical extension* of H_Φ from the boundary component B_Φ to the symmetric space M .

By construction we have $\mathfrak{h}_\Phi^\Lambda \cap \mathfrak{k} = \mathfrak{h}_\Phi \cap \mathfrak{k}$, and the normal space at o in $T_o M$ of the orbit $H_\Phi^\Lambda \cdot o$ coincides with the normal space at o in $T_o B_\Phi$ of the orbit $H_\Phi \cdot o$. This implies that the slice representations of H_Φ^Λ and H_Φ at o coincide. Therefore, the cohomogeneity of the action of H_Φ^Λ on M coincides with the cohomogeneity of the action of H_Φ on B_Φ . We therefore conclude:

Proposition 4.1. *Let M be a connected Riemannian symmetric space of noncompact type and let B_Φ be a boundary component of M . Then every cohomogeneity one action on B_Φ has a canonical extension to a cohomogeneity one action on M .*

Remark. If $\Phi \subset \Psi \subset \Lambda$ are nonempty proper subsets of each other, then clearly $B_\Phi \subset B_\Psi \subset M$ are proper totally geodesic submanifolds of each other, and B_Φ is a boundary component of the semisimple symmetric space B_Ψ . Let H_Φ be a connected subgroup of $I(B_\Phi)$ acting on B_Φ with cohomogeneity one, and denote by H_Φ^Ψ the connected subgroup of $I(B_\Psi)$ obtained by canonical extension of the H_Φ -action from B_Φ to B_Ψ , by H_Φ^Λ the connected subgroup of $I(M)$ obtained by canonical extension of the H_Φ -action from B_Φ to M , and by H_Ψ^Λ the connected subgroup of $I(M)$ obtained by canonical extension of the H_Ψ^Ψ -action from B_Ψ to M . It follows from the construction that the corresponding Lie algebras satisfy $\mathfrak{h}_\Phi^\Lambda = \mathfrak{h}_\Psi^\Lambda$, and therefore $H_\Phi^\Lambda = H_\Psi^\Lambda$ as both groups are connected. This shows that for the classification of cohomogeneity one actions on M obtained by canonical extensions one can restrict to canonical extensions of cohomogeneity one actions on boundary components B_Φ of rank $r-1$, that is, those boundary components B_Φ obtained from subsets Φ of Λ with $|\Phi| = r-1$.

We now investigate in how far canonical extensions preserve orbit equivalence of cohomogeneity one actions.

Proposition 4.2. *Let M be a connected Riemannian symmetric space of noncompact type and let B_Φ be a boundary component of M . Let H_Φ^1, H_Φ^2 be two connected closed subgroups of $I(B_\Phi)$ which act on B_Φ with cohomogeneity one. Assume that these two actions are orbit equivalent by an isometry $f \in I^o(B_\Phi)$. Then the two cohomogeneity one actions on M which are obtained by canonical extension of H_Φ^1 and H_Φ^2 are orbit equivalent.*

Proof. Since G_Φ is a connected semisimple Lie group acting transitively on B_Φ , we must have $I^o(B_\Phi) \subset G_\Phi$. Since $G_\Phi \subset M_\Phi \subset L_\Phi \subset Q_\Phi \subset G$, the isometry f extends canonically to an isometry F in the parabolic subgroup Q_Φ of G . The horospherical decomposition

$M = B_\Phi \times A_\Phi \times N_\Phi$ shows that

$$F((H_\Phi^1)^\Lambda \cdot p) = f(H_\Phi^1 \cdot p) \times A_\Phi \times N_\Phi.$$

By assumption, we have

$$f(H_\Phi^1 \cdot p) \times A_\Phi \times N_\Phi = (H_\Phi^2 \cdot f(p)) \times A_\Phi \times N_\Phi.$$

Recall that we have an analytic diffeomorphism $M_\Phi \times A_\Phi \times N_\Phi \rightarrow Q_\Phi$, and accordingly we write $F = (\bar{m}, 1, 1)$ with $\bar{m} \in G_\Phi$. Since the group structure is given by

$$(m, a, n)(m', a', n') = (mm', aa', (m'a')^{-1}n(m'a')n'),$$

we obtain

$$\begin{aligned} (H_\Phi^2)^\Lambda \cdot F(p) &= \{(m, a, n)(\bar{m}, 1, 1) \cdot p \mid m \in H_\Phi^2, a \in A_\Phi, n \in N_\Phi\} \\ &= \{(m\bar{m}, a, \bar{m}^{-1}n\bar{m}) \cdot p \mid m \in H_\Phi^2, a \in A_\Phi, n \in N_\Phi\} \\ &= \{(m\bar{m}, a, n) \cdot p \mid m \in H_\Phi^2, a \in A_\Phi, n \in N_\Phi\} \\ &= (H_\Phi^2 \cdot f(p)) \times A_\Phi \times N_\Phi. \end{aligned}$$

Altogether we see that $F((H_\Phi^1)^\Lambda \cdot p) = (H_\Phi^2)^\Lambda \cdot F(p)$, which means that the two cohomogeneity one actions on M obtained by canonical extension of H_Φ^1 and H_Φ^2 are orbit equivalent. q.e.d.

The following example shows that we cannot weaken the assumption in Proposition 4.2 from $f \in I^o(B_\Phi)$ to $f \in I(B_\Phi)$.

Example. We consider the symmetric space $M = SL_4(\mathbb{R})/SO_4$. This symmetric space has rank 3, dimension 9, and the restricted root system is of type (A_3) with all multiplicities equal to one. We choose $\Phi = \{\alpha_1, \alpha_2\} \subset \Lambda = \{\alpha_1, \alpha_2, \alpha_3\}$. Then we have

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \mathfrak{g}_{\pm\alpha_1} \oplus \mathfrak{g}_{\pm\alpha_2} \oplus \mathfrak{g}_{\pm(\alpha_1+\alpha_2)} \cong \mathfrak{sl}_3(\mathbb{R}) \oplus \mathbb{R},$$

and therefore

$$F_\Phi = L_\Phi \cdot o \cong SL_3(\mathbb{R})/SO_3 \times \mathbb{E}.$$

The corresponding boundary component B_Φ is isometric to $SL_3(\mathbb{R})/SO_3$. We now define two subalgebras \mathfrak{h}_Φ^1 and \mathfrak{h}_Φ^2 of $\mathfrak{sl}_3(\mathbb{R})$ by

$$\mathfrak{h}_\Phi^1 = \mathfrak{a}^\Phi \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \quad \text{and} \quad \mathfrak{h}_\Phi^2 = \mathfrak{a}^\Phi \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}.$$

The corresponding connected subgroups H_Φ^1 and H_Φ^2 of $SL_3(\mathbb{R})$ act on the boundary component B_Φ with cohomogeneity one, and the orbits form a foliation on B_Φ . These two actions are orbit equivalent, and the corresponding isometry is induced by the Dynkin diagram symmetry of (A_2) , the restricted root system of B_Φ . We now consider the canonical extensions of these two actions, which are defined by

$$(\mathfrak{h}_\Phi^1)^\Lambda = \mathfrak{h}_\Phi^1 \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi \quad \text{and} \quad (\mathfrak{h}_\Phi^2)^\Lambda = \mathfrak{h}_\Phi^2 \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi.$$

In terms of the root space decomposition of M these two subalgebras are

$$(\mathfrak{h}_\Phi^1)^\Lambda = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_1}) \quad \text{and} \quad (\mathfrak{h}_\Phi^2)^\Lambda = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_2})$$

The corresponding connected subgroups $(H_\Phi^1)^\Lambda$ and $(H_\Phi^2)^\Lambda$ of $SL_4(\mathbb{R})$ act on the symmetric space $M = SL_4(\mathbb{R})/SO_4$ with cohomogeneity one, and the orbits form a foliation on M . However, these two actions are not orbit equivalent since there is no corresponding Dynkin diagram symmetry (see [3] for details). The reason for this is that the Dynkin diagram symmetry of (A_2) does not extend to a Dynkin diagram symmetry of (A_3) .

4.2. Nilpotent construction. We now describe our second new method for constructing cohomogeneity one actions on M .

Let Φ be a subset of Λ and consider the parabolic subalgebra \mathfrak{q}_Φ and its Langlands decomposition $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$. Recall that $\mathfrak{q}_\Phi \cap \mathfrak{k} = \mathfrak{m}_\Phi \cap \mathfrak{k} = \mathfrak{k}_\Phi$, and we have a canonical isomorphism

$$T_o M \cong \mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi, \quad T_o F_\Phi \cong \mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi, \quad T_o B_\Phi \cong \mathfrak{b}_\Phi.$$

Since $K_\Phi \subset L_\Phi = Z_G(\mathfrak{a}_\Phi)$, we have $\text{Ad}(k)X = X$ for all $k \in K_\Phi$ and $X \in \mathfrak{a}_\Phi$. Furthermore, since $K_\Phi \subset M_\Phi$ and $\mathfrak{m}_\Phi = \mathfrak{k}_\Phi \oplus \mathfrak{b}_\Phi$ is a Cartan decomposition, we get $\text{Ad}(k)(\mathfrak{b}_\Phi) = \mathfrak{b}_\Phi$ for all $k \in K_\Phi$. Eventually, since $K_\Phi \subset L_\Phi$ normalizes N_Φ , we get $\text{Ad}(k)(\mathfrak{n}_\Phi) = \mathfrak{n}_\Phi$ for all $k \in K_\Phi$. Thus the decomposition $T_o M \cong \mathfrak{b}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ is $\text{Ad}(K_\Phi)$ -invariant.

The nilpotent subalgebra \mathfrak{n}_Φ has a natural gradation which we shall now describe. Let $H^1, \dots, H^r \in \mathfrak{a}$ be the dual vectors of $\alpha_1, \dots, \alpha_r$, that is, define $H^1, \dots, H^r \in \mathfrak{a}$ by $\alpha_\nu(H^\mu) = \delta_{\nu\mu}$. Define

$$H^\Phi = \sum_{\alpha_i \in \Lambda \setminus \Phi} H^i$$

and $m_\Phi = \tilde{\alpha}(H^\Phi)$, where $\tilde{\alpha}$ is the highest root in Σ^+ . For each $\alpha \in \Sigma^+$ we have $\alpha(H^\Phi) \in \{0, \dots, m_\Phi\}$, and $\alpha \in \Sigma_\Phi^+$ if and only if $\alpha(H^\Phi) = 0$. For each $\nu \in \{1, \dots, m_\Phi\}$ we define a subspace \mathfrak{n}_Φ^ν of \mathfrak{n}_Φ by

$$\mathfrak{n}_\Phi^\nu = \bigoplus_{\substack{\alpha \in \Sigma^+ \setminus \Sigma_\Phi^+ \\ \alpha(H^\Phi) = \nu}} \mathfrak{g}_\alpha.$$

Then

$$\mathfrak{n}_\Phi = \bigoplus_{\nu=1}^{m_\Phi} \mathfrak{n}_\Phi^\nu$$

is an $\text{Ad}(K_\Phi)$ -invariant gradation of \mathfrak{g} . This gradation is generated by \mathfrak{n}_Φ^1 , which means that $[\mathfrak{n}_\Phi^1, \mathfrak{n}_\Phi^\nu] = \mathfrak{n}_\Phi^{\nu+1}$ holds for all $\nu \in \{1, \dots, m_\Phi - 1\}$ (see [15]). It is clear that \mathfrak{n}_Φ is abelian if and only if $m_\Phi = 1$.

Assume that $\dim \mathfrak{n}_\Phi^1 \geq 2$ and let \mathfrak{v} be a subspace of \mathfrak{n}_Φ^1 with $\dim \mathfrak{v} \geq 2$. Since $[\mathfrak{n}_\Phi, \mathfrak{n}_\Phi] = \mathfrak{n}_\Phi \ominus \mathfrak{n}_\Phi^1$, we see that

$$\mathfrak{n}_{\Phi, \mathfrak{v}} = \mathfrak{n}_\Phi \ominus \mathfrak{v}$$

is a subalgebra of \mathfrak{n}_Φ . Let $N_{\Phi, \mathfrak{v}}$ be the connected subgroup of N_Φ with Lie algebra $\mathfrak{n}_{\Phi, \mathfrak{v}}$ and $N_{L_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ be the normalizer of $\mathfrak{n}_{\Phi, \mathfrak{v}}$ in L_Φ . Since $K_\Phi = L_\Phi \cap K$, the normalizer $N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ of $\mathfrak{n}_{\Phi, \mathfrak{v}}$ in K_Φ coincides with $N_{L_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}}) \cap K$. Moreover, since $\text{Ad}(k)$ acts as an orthogonal transformation on \mathfrak{n}_Φ for each $k \in K_\Phi$, $N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ coincides with the normalizer $N_{K_\Phi}(\mathfrak{v})$ of \mathfrak{v} in K_Φ . Denote by $N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}})$ and $N_{K_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}})$ the connected component of $N_{L_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ and $N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}})$ containing the identity transformation on M , respectively. Then $H_{\Phi, \mathfrak{v}} = N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}})N_{\Phi, \mathfrak{v}}$ is a connected subgroup of Q_Φ . Assume that $N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}})$ acts transitively on F_Φ , which just means that $F_\Phi \subset H_{\Phi, \mathfrak{v}} \cdot o$. Since $H_{\Phi, \mathfrak{v}} \cap K = N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}}) \cap K$ and $N_{L_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}}) \cap K = N_{K_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}}) = N_{K_\Phi}(\mathfrak{v})$, we see that the cohomogeneity of the action of $H_{\Phi, \mathfrak{v}}$ on M is equal to the cohomogeneity of the action of $N_{K_\Phi}^o(\mathfrak{v})$ on \mathfrak{v} . Since L_Φ is reductive, we also have $N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}}) = \theta N_{L_\Phi}^o(\mathfrak{v})$. Thus we get the following construction method for cohomogeneity one actions on M .

Proposition 4.3. *Assume that $\dim \mathfrak{n}_\Phi^1 \geq 2$ and let \mathfrak{v} be a subspace of \mathfrak{n}_Φ^1 with $\dim \mathfrak{v} \geq 2$ such that*

- (i) $N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}}) = \theta N_{L_\Phi}^o(\mathfrak{v})$ acts transitively on F_Φ , and
- (ii) $N_{K_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}}) = N_{K_\Phi}^o(\mathfrak{v})$ acts transitively on the unit sphere in \mathfrak{v} .

Then $H_{\Phi, \mathfrak{v}} = N_{L_\Phi}^o(\mathfrak{n}_{\Phi, \mathfrak{v}})N_{\Phi, \mathfrak{v}}$ acts on M with cohomogeneity one and $H_{\Phi, \mathfrak{v}} \cdot o$ is a singular orbit of this action containing F_Φ . Moreover, if \mathfrak{v}_1 and \mathfrak{v}_2 are two such subspaces which are conjugate by an element in K_Φ , then the cohomogeneity one actions of H_{Φ, \mathfrak{v}_1} and H_{Φ, \mathfrak{v}_2} on M are orbit equivalent.

Proof. We only have to prove the statement about orbit equivalence. Assume that $\text{Ad}(k)(\mathfrak{v}_1) = \mathfrak{v}_2$ for some $k \in K_\Phi$. Since $\text{Ad}(k)$ preserves the Chevalley decomposition $\mathfrak{q}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi$, we have $\text{Ad}(k)(N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1}) \oplus \mathfrak{n}_{\Phi, \mathfrak{v}_1}) = \text{Ad}(k)N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1}) \oplus \text{Ad}(k)\mathfrak{n}_{\Phi, \mathfrak{v}_1}$. By assumption, we have $\text{Ad}(k)(\mathfrak{v}_1) = \mathfrak{v}_2$, and since $\text{Ad}(k)$ acts as an orthogonal transformation on \mathfrak{n}_Φ , this implies $\text{Ad}(k)\mathfrak{n}_{\Phi, \mathfrak{v}_1} = \mathfrak{n}_{\Phi, \mathfrak{v}_2}$. If $X \in N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1})$, we get

$$[\text{Ad}(k)X, \mathfrak{n}_{\Phi, \mathfrak{v}_2}] = [\text{Ad}(k)X, \text{Ad}(k)\mathfrak{n}_{\Phi, \mathfrak{v}_1}] = \text{Ad}(k)[X, \mathfrak{n}_{\Phi, \mathfrak{v}_1}] \subset \text{Ad}(k)\mathfrak{n}_{\Phi, \mathfrak{v}_1} = \mathfrak{n}_{\Phi, \mathfrak{v}_2},$$

which implies $\text{Ad}(k)N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1}) \subset N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_2})$. By an analogous argumentation we obtain $\text{Ad}(k^{-1})N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_2}) \subset N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1})$. Altogether this shows that $\text{Ad}(k)N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_1}) = N_{\mathfrak{l}_\Phi}(\mathfrak{n}_{\Phi, \mathfrak{v}_2})$, and therefore $\text{Ad}(k)\mathfrak{h}_{\Phi, \mathfrak{v}_1} = \mathfrak{h}_{\Phi, \mathfrak{v}_2}$. Since both H_{Φ, \mathfrak{v}_1} and H_{Φ, \mathfrak{v}_2} are connected, this implies that the actions of H_{Φ, \mathfrak{v}_1} and H_{Φ, \mathfrak{v}_2} are orbit equivalent. q.e.d.

We now discuss the second construction method in more detail for maximal proper parabolic subgroups. Any such subgroup is conjugate to Q_{Φ_j} for some $j \in \{1, \dots, r\}$, where $\Phi_j = \Lambda \setminus \{\alpha_j\}$. In the following we will replace the “index Φ_j ” by the “index j ”, that is, the parabolic subalgebra \mathfrak{q}_{Φ_j} will be denoted by \mathfrak{q}_j , and so on. We discuss now a few examples of cohomogeneity one actions arising from this construction method.

Examples. 1. Assume that the rank of M is equal to one. Thus M is isometric to a hyperbolic space $\mathbb{F}H^n$ over a normed real division algebra $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. In this case there is just one simple root $\alpha = \alpha_1$, and therefore $\Phi_1 = \emptyset$. The maximal proper parabolic subgroup Q_1 is therefore a minimal parabolic subgroup. The parabolic subalgebra \mathfrak{q}_1 is given by $\mathfrak{q}_1 = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. More explicitly, we have the following table:

M	G	K	K_0	\mathfrak{g}_α	\mathfrak{n}
$\mathbb{R}H^n$	$SO_{1,n}^o$	SO_n	SO_{n-1}	\mathbb{R}^{n-1}	\mathbb{R}^{n-1}
$\mathbb{C}H^n$	$SU_{1,n}$	U_n	U_{n-1}	\mathbb{C}^{n-1}	$\mathbb{C}^{n-1} \oplus \mathbb{R}$
$\mathbb{H}H^n$	$Sp_{1,n}$	$Sp_1 Sp_n$	$Sp_1 Sp_{n-1}$	\mathbb{H}^{n-1}	$\mathbb{H}^{n-1} \oplus \mathbb{R}^3$
$\mathbb{O}H^2$	F_4^{-20}	$Spin_9$	$Spin_7$	\mathbb{O}	$\mathbb{O} \oplus \mathbb{R}^7$

Condition (i) in Proposition 4.3 is automatically satisfied since the boundary component B_1 consists of the single point o . Condition (ii) is equivalent to the problem: Find all k -dimensional ($k \geq 2$) linear subspaces \mathfrak{v} of \mathfrak{g}_α for which there exists a subgroup of K_0 acting transitively on the unit sphere in \mathfrak{v} . The authors solved this problem in [5] for $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{O}\}$, whereas for $\mathbb{F} = \mathbb{H}$ we only found some examples but achieved no complete classification.

If $\mathbb{F} = \mathbb{R}$, we can choose any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$. However, in this case the orbit $H_{1, \mathfrak{v}} \cdot o$ is always totally geodesic in $\mathbb{R}H^n$.

If $\mathbb{F} = \mathbb{C}$, we can choose any linear subspace $\mathfrak{v} \subset \mathbb{C}^{n-1}$ with constant Kähler angle $\varphi \in [0, \pi/2]$. If $0 < \varphi < \pi/2$, then the cohomogeneity one action on $\mathbb{C}H^n$ by $H_{1, \mathfrak{v}}$ has a non-totally geodesic singular orbit and is not orbit equivalent to a cohomogeneity one action obtained by any of the other construction methods.

If $\mathbb{F} = \mathbb{O}$, we can choose any linear subspace $\mathfrak{v} \subset \mathbb{O}$ of dimension $k \in \{2, 3, 4, 6, 7\}$. The cohomogeneity one action on $\mathbb{O}H^2$ by $H_{1,\mathfrak{v}}$ has a non-totally geodesic singular orbit and is not orbit equivalent to a cohomogeneity one action obtained by any of the other construction methods.

If $\mathbb{F} = \mathbb{H}$, we can choose linear subspaces $\mathfrak{v} \subset \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle. However, the classification of such subspaces is not yet finalized.

2. Let $M = G_2^2/SO_4$. Then $\dim M = 8$, and $\mathfrak{g} = \mathfrak{g}_2^2$ is a split real form of $\mathfrak{g}^{\mathbb{C}}$. For $M = G_2^2/SO_4$ the corresponding root system Σ is of type (G_2) and all root spaces have real dimension 1. We label the simple roots by α_1 and α_2 so that $3\alpha_1 + 2\alpha_2$ is the highest root in Σ^+ , and choose $j = 1$, that is, $\Phi_1 = \{\alpha_2\}$. Then we have $\mathfrak{k}_0 = \{0\}$ and $\mathfrak{g}_0 = \mathfrak{a} \cong \mathbb{R}^2$. Moreover,

$$\begin{aligned} \mathfrak{n}_1^1 &= \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2, \\ \mathfrak{n}_1^2 &= \mathfrak{g}_{2\alpha_1+\alpha_2} \cong \mathbb{R}, \\ \mathfrak{n}_1^3 &= \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{R}^2, \\ \mathfrak{n}_1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_1^2 \oplus \mathfrak{n}_1^3 &= \mathfrak{n} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{R}^5, \\ \mathfrak{a}_1 &= \ker \alpha_2 = \mathbb{R}H^1 \cong \mathbb{R}, \\ \mathfrak{m}_1 = \mathfrak{g}_1 &= \mathfrak{g}_{-\alpha_2} \oplus \mathbb{R}H^2 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{sl}_2(\mathbb{R}), \\ \mathfrak{l}_1 = \mathfrak{m}_1 \oplus \mathfrak{a}_1 &= \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}, \\ \mathfrak{k}_1 &= \mathfrak{k}_{\alpha_2} \cong \mathfrak{so}_2. \end{aligned}$$

This explicit description shows that $F_1 \cong SL_2(\mathbb{R})/SO_2 \times \mathbb{E} = \mathbb{R}H^2 \times \mathbb{E}$ and that $K_1^o \cong SO_2$ acts transitively on the unit sphere in $\mathfrak{v} = \mathfrak{n}_1^1 \cong \mathbb{R}^2$. It follows that $H_{1,\mathfrak{v}}$ acts on M with cohomogeneity one whose singular orbit has codimension 2 and contains $F_1 \cong \mathbb{R}H^2 \times \mathbb{E}$. The Lie algebra of $H_{1,\mathfrak{v}}$ is given by

$$\mathfrak{h}_{1,\mathfrak{v}} = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2}.$$

3. Let $M = G_2^{\mathbb{C}}/G_2$. Then $\dim M = 14$, and the corresponding root system Σ is of type (G_2) and can be identified with the root system of the complex simple Lie algebra $(\mathfrak{g}_2)^{\mathbb{C}}$. Therefore all root spaces have complex dimension 1. As in the previous example we label the simple roots by α_1 and α_2 so that $3\alpha_1 + 2\alpha_2$ is the highest root in Σ^+ , and choose again $j = 1$, and hence $\Phi_1 = \{\alpha_2\}$. Then we have $\mathfrak{k}_0 \cong \mathfrak{u}_1 \oplus \mathfrak{u}_1$, $\mathfrak{g}_0 \cong \mathfrak{u}_1 \oplus \mathfrak{u}_1 \oplus \mathbb{R}^2$, and

$$\begin{aligned} \mathfrak{n}_1^1 &= \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{C}^2, \\ \mathfrak{n}_1^2 &= \mathfrak{g}_{2\alpha_1+\alpha_2} \cong \mathbb{C}, \\ \mathfrak{n}_1^3 &= \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{C}^2, \\ \mathfrak{n}_1 = \mathfrak{n}_1^1 \oplus \mathfrak{n}_1^2 \oplus \mathfrak{n}_1^3 &= \mathfrak{n} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} \cong \mathbb{C}^5, \\ \mathfrak{l}_1 &= \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{u}_1 \oplus \mathbb{R}, \\ \mathfrak{a}_1 &= \ker \alpha_2 = \mathbb{R}H^1 \cong \mathbb{R}, \\ \mathfrak{m}_1 = \mathfrak{l}_1 \ominus \mathfrak{a}_1 &\cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{u}_1, \\ \mathfrak{g}_1 = [\mathfrak{m}_1, \mathfrak{m}_1] &\cong \mathfrak{sl}_2(\mathbb{C}), \\ \mathfrak{z}_1 &\cong \mathfrak{u}_1, \\ \mathfrak{k}_1 &= \mathfrak{k}_{\alpha_2} \oplus \mathfrak{u}_1 \cong \mathfrak{su}_2 \oplus \mathfrak{u}_1 \cong \mathfrak{u}_2. \end{aligned}$$

From this we see that $F_1 \cong SL_2(\mathbb{C})/SU_2 \times \mathbb{E} \cong \mathbb{R}H^3 \times \mathbb{E}$. Moreover, $K_1^o \cong U_2$ acts transitively on the unit sphere in $\mathfrak{v} = \mathfrak{n}_1^1 \cong \mathbb{C}^2$. It follows that $H_{1,\mathfrak{v}}$ acts on M with cohomogeneity one whose singular orbit has codimension 4 and contains $F_1 \cong \mathbb{R}H^3 \times \mathbb{E}$. The Lie algebra of $H_{1,\mathfrak{v}}$ is given by

$$\mathfrak{h}_{1,\mathfrak{v}} = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2}.$$

4. The following example illustrates that the two different construction methods can lead to orbit equivalent cohomogeneity one actions even when $|\Lambda \setminus \Phi| = 1$. Let $M = SO_{2,n+2}^o/SO_2SO_{n+2}$ and $n \geq 1$. Then $\dim M = 2n+4$ and the corresponding root system Σ is of type (B_2) . Let α_1 and α_2 be corresponding simple roots such that α_1 is the longer of the two roots. Then we have $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, and the multiplicities of the two long roots α_1 and $\alpha_1 + 2\alpha_2$ are 1 and of the two short roots α_2 and $\alpha_1 + \alpha_2$ are n . We have $\mathfrak{k}_0 \cong \mathfrak{so}_n$ and $\mathfrak{a} \cong \mathbb{R}^2$.

Firstly, we choose $j = 1$, that is, $\Phi_1 = \{\alpha_2\}$. Then we have

$$\begin{aligned} \mathfrak{n}_1 &= \mathfrak{n}_1^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R}^{n+2}, \\ \mathfrak{l}_1 &= \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{so}_{1,n+1} \oplus \mathbb{R}, \\ \mathfrak{a}_1 &= \ker \alpha_2 = \mathbb{R}H^1 \cong \mathbb{R}, \\ \mathfrak{g}_1 &= \mathfrak{m}_1 = \mathfrak{l}_1 \ominus \mathfrak{a}_1 \cong \mathfrak{so}_{1,n+1} \\ \mathfrak{k}_1 &= \mathfrak{so}_{n+1}. \end{aligned}$$

From this we see that $F_1 \cong SO_{1,n+1}^o/SO_{n+1} \times \mathbb{E}$ and $K_1^o \cong SO_{n+1}$. The K_1^o -module \mathfrak{n}_1 decomposes into a 1-dimensional trivial module $\mathfrak{n}_{1,0} \cong \mathbb{R} \subset \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R}^2$ and an irreducible module $\mathfrak{v} \cong \mathbb{R}^{n+1} \supset \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^n$. The action of $K_1^o \cong SO_{n+1}$ on the irreducible module $\mathfrak{v} \cong \mathbb{R}^{n+1}$ is the standard one and acts transitively on the unit sphere. It follows that $H_{1,\mathfrak{v}}$ acts on M with cohomogeneity one whose singular orbit W has codimension $n+1$ and contains $F_1 \cong \mathbb{R}H^{n+1} \times \mathbb{E}$. The Lie algebra of $H_{1,\mathfrak{v}}$ is given by

$$\mathfrak{h}_{1,\mathfrak{v}} = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{n}_{1,0} = \mathfrak{g}_1 \oplus (\mathfrak{a}_1 \oplus \mathfrak{n}_{1,0}).$$

However, the orbit through o of the connected subgroup of G with Lie algebra $\mathfrak{a}_1 \oplus \mathfrak{n}_{1,0}$ is a totally geodesic real hyperbolic plane $\mathbb{R}H^2$, and hence W is a totally geodesic submanifold of M which is congruent to the Riemannian product $\mathbb{R}H^{n+1} \times \mathbb{R}H^2$ of two real hyperbolic spaces.

Finally, we choose $j = 2$, that is, $\Phi_2 = \{\alpha_1\}$. Then we have

$$\begin{aligned} \mathfrak{n}_2^1 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^{2n}, \\ \mathfrak{n}_2^2 &= \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R} \\ \mathfrak{n}_2 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R}^{2n+1} \\ \mathfrak{l}_2 &= \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{so}_{1,2} \oplus \mathfrak{so}_n \oplus \mathbb{R}, \\ \mathfrak{a}_2 &= \ker \alpha_1 = \mathbb{R}H^2 \cong \mathbb{R}, \\ \mathfrak{g}_2 &= \mathfrak{m}_2 = \mathfrak{l}_2 \ominus \mathfrak{a}_2 \cong \mathfrak{so}_{1,2} \oplus \mathfrak{so}_n \\ \mathfrak{k}_2 &\cong \mathfrak{so}_2 \oplus \mathfrak{so}_n. \end{aligned}$$

From this we see that $F_2 \cong SO_{1,2}^o/SO_2 \times \mathbb{E} = \mathbb{R}H^2 \times \mathbb{E}$ and $K_2^o \cong SO_2SO_n$. The representation of K_2^o on \mathfrak{n}_2^1 is isomorphic to the tensor representation of SO_2SO_n on $\mathbb{R}^2 \otimes \mathbb{R}^n \cong \mathbb{R}^{2n}$.

The symmetric space $M = SO_{2,n+2}^o/SO_2SO_{n+2}$ is Hermitian and hence has a natural complex structure J . This complex structure turns $\mathfrak{n}_2^1 \cong \mathbb{R}^{2n}$ into a complex vector space \mathbb{C}^n so that \mathfrak{g}_{α_2} and $\mathfrak{g}_{\alpha_1+\alpha_2}$ are real subspaces which are mapped onto each other by J . Moreover, the action of $SO_2 \subset SO_2SO_n \cong K_2^o$ on \mathfrak{n}_2^1 is isomorphic to the standard action of the circle group on \mathbb{C}^n , and the action of $SO_n \subset SO_2SO_n \cong K_2^o$ on $\mathbb{R}^n \cong \mathfrak{g}_{\alpha_2} \subset \mathfrak{n}_2^1$ and on $\mathbb{R}^n \cong \mathfrak{g}_{\alpha_1+\alpha_2} \subset \mathfrak{n}_2^1$ is isomorphic to the standard action of SO_n on \mathbb{R}^n . We now construct cohomogeneity one actions on M through two different types of subspaces \mathfrak{v} .

Firstly, let \mathfrak{v} be a k -dimensional linear subspace of \mathfrak{g}_{α_2} with $k \geq 2$. Then $N_{K_2^o}(\mathfrak{v})$ is isomorphic to $SO_kSO_{n-k} \subset SO_n \subset SO_2SO_n$ and acts transitively on the unit sphere in \mathfrak{v} . Moreover,

$$N_{\mathfrak{l}_2}(\mathfrak{n}_{2,\mathfrak{v}}) = (\mathfrak{so}_k \oplus \mathfrak{so}_{n-k}) \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1}$$

where $\mathfrak{so}_k \oplus \mathfrak{so}_{n-k} \cong N_{\mathfrak{k}_2}(\mathfrak{v})$. We easily see that the connected subgroup of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{g}_{\alpha_1}$ acts transitively on $F_2 \cong SO_{1,2}^o/SO_2 \times \mathbb{E}$. Altogether it follows that $H_{2,\mathfrak{v}}$ acts on M with cohomogeneity one and with a singular orbit W of codimension k and containing F_2 . The Lie algebra of $H_{2,\mathfrak{v}}$ is given by

$$\mathfrak{h}_{2,\mathfrak{v}} = (\mathfrak{so}_k \oplus \mathfrak{so}_{n-k}) \oplus \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1} \oplus (\mathfrak{g}_{\alpha_2} \ominus \mathfrak{v}) \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}.$$

Note that $\mathfrak{so}_k \oplus \mathfrak{so}_{n-k} \subset \mathfrak{so}_n \cong \mathfrak{k}_0$. However, it is evident from the explicit description of $\mathfrak{h}_{2,\mathfrak{v}}$ that the action of $H_{2,\mathfrak{v}}$ on M is orbit equivalent to the action of the canonical extension of a cohomogeneity one action on the boundary component $B_1 = SO_{1,n+1}^o/SO_{n+1}$. Instead of picking a subspace \mathfrak{v} of the real subspace \mathfrak{g}_{α_2} of \mathfrak{n}_2^1 , we could also select a subspace \mathfrak{v} of any of the real subspaces of \mathfrak{n}_2^1 obtained by rotating \mathfrak{g}_{α_2} in \mathfrak{n}_2^1 by means of the SO_2 -action with the SO_2 whose Lie algebra is \mathfrak{so}_2 in $\mathfrak{k}_2 \cong \mathfrak{so}_2 \oplus \mathfrak{so}_n$. For example, $\mathfrak{g}_{\alpha_1+\alpha_2}$ is such a subspace. However, such a cohomogeneity one action is conjugate to one constructed from a subspace in \mathfrak{g}_{α_2} .

5. PROOF OF THEOREM 1.1

Let H be a connected subgroup of G acting on M with cohomogeneity one. If the orbits form a Riemannian foliation, a complete classification up to orbit equivalence was obtained by the authors in [3] for irreducible symmetric spaces M . For reducible symmetric spaces the corresponding problem is still unsolved. We assume from now on that the action has a singular orbit W . Then H is contained either in a proper maximal reductive subgroup of G or in a proper maximal parabolic subgroup of G . In the first case we have a totally geodesic singular orbit (see Theorem 3.2). For irreducible symmetric spaces M the classification of such actions was obtained by the authors in [4]. For reducible symmetric spaces the corresponding problem is still not solved. We assume from now on that H is contained in a proper maximal parabolic subgroup of G , or equivalently, $\mathfrak{h} \subset \mathfrak{q}_j$ for some $j \in \{1, \dots, r\}$. Without loss of generality we may assume that $o \in W$, that is $W = H \cdot o$.

Consider the slice representation

$$\chi : H \cap K \rightarrow O(\nu_o W), \quad k \mapsto d_o k|_{\nu_o W}.$$

Since $H \subset Q_j$, we have $H \cap K \subset Q_j \cap K = K_j$, and therefore $d_o k(\xi) = \text{Ad}(k)\xi$ for all $\xi \in \nu_o W$ and $k \in H \cap K$, where we identify

$$T_o M \cong (\mathfrak{l}_j \cap \mathfrak{p}) \oplus \mathfrak{n}_j = \mathfrak{b}_j \oplus \mathfrak{a}_j \oplus \left(\bigoplus_{\nu=1}^{m_j} \mathfrak{n}_j^\nu \right).$$

Recall that $T_o B_j \cong \mathfrak{b}_j$ under the above identification.

We first show that the normal space $\nu_o W$ is contained in either $\mathfrak{b}_j \cong T_o B_j$ or \mathfrak{n}_j^1 . First of all, we use the fact that $H \cap K = H \cap K_j$ acts transitively on the unit sphere in $\nu_o W$. We decompose the parabolic subalgebra \mathfrak{q}_j into $\mathfrak{q}_j = \mathfrak{k}_j \oplus \mathfrak{b}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$ and denote by $\tau : \mathfrak{q}_j \rightarrow \mathfrak{b}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$ the canonical projection with respect to this decomposition. Then we have $\tau(\mathfrak{h}) = T_o W = (\mathfrak{b}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j) \ominus \nu_o W$. Since $\dim \mathfrak{a}_j = 1$, we must have

$$\nu_o W \subset \mathfrak{b}_j \oplus \mathfrak{n}_j. \quad (5.1)$$

Let us define

$$\begin{aligned} (\nu_o W)_0 &:= \mathfrak{b}_j \ominus (\tau(\mathfrak{h}) \cap \mathfrak{b}_j), \\ (\nu_o W)_\nu &:= \mathfrak{n}_j^\nu \ominus (\tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu) \quad \text{for } \nu = 1, \dots, m_j. \end{aligned}$$

It is easy to see that

$$\nu_o W \subset (\nu_o W)_0 \oplus (\nu_o W)_1 \oplus \dots \oplus (\nu_o W)_{m_j}. \quad (5.2)$$

Lemma 5.1. *Let $0 \neq X \in \nu_o W$, and denote by $\pi_k : (\nu_o W)_0 \oplus \dots \oplus (\nu_o W)_{m_j} \rightarrow (\nu_o W)_k$ the canonical projection. Then we have*

- (1) $\nu_o W = \mathbb{R}X \oplus [\mathfrak{h} \cap \mathfrak{k}_j, X]$.
- (2) $\pi_k|_{\nu_o W} : \nu_o W \rightarrow (\nu_o W)_k$ is onto and $(H \cap K_j)$ -equivariant.
- (3) If $\pi_k(X) = 0$ then $(\nu_o W)_k = 0$.
- (4) If $(\nu_o W)_k \neq 0$ then $\pi_k|_{\nu_o W} : \nu_o W \rightarrow (\nu_o W)_k$ is an $(H \cap K_j)$ -equivariant isomorphism. In particular, $(\nu_o W)_k$ is an irreducible $(\mathfrak{h} \cap \mathfrak{k}_j)$ -module.

Proof. Since $H \cap K_j$ acts transitively on the unit sphere in $\nu_o W$, the subspace $[\mathfrak{h} \cap \mathfrak{k}_j, X]$ in $\nu_o W$ has codimension one. This implies (1) since $[\mathfrak{h} \cap \mathfrak{k}_j, X]$ is perpendicular to X . Statement (2) follows from the fact that $H \cap K_j$ preserves the decomposition (5.2). To show (3), assume that $\pi_k(X) = 0$. This means $\langle X, (\nu_o W)_k \rangle = 0$. Since $\mathfrak{h} \cap \mathfrak{k}_j$ normalizes $(\nu_o W)_k$ and by (1), we have $\langle \nu_o W, (\nu_o W)_k \rangle = 0$. This implies $(\nu_o W)_k \subset \tau(\mathfrak{h})$ and hence $(\nu_o W)_k = 0$. To show (4), assume that $(\nu_o W)_k \neq 0$. Then $\pi_k|_{\nu_o W}$ is injective by (3), and taking into account (2), we see that $\pi_k|_{\nu_o W}$ is an isomorphism. q.e.d.

In the second step we use $\mathfrak{a}_j \subset \tau(\mathfrak{h})$, which follows directly from (5.1). By definition, there exists $H_\mathfrak{k}^j \in \mathfrak{k}_j$ such that $H_\mathfrak{k}^j + H^j \in \mathfrak{h}$. Note that $H_\mathfrak{k}^j \in (\mathfrak{h})_{\mathfrak{k}_j}$, where $(\mathfrak{h})_{\mathfrak{k}_j}$ is obtained by orthogonally projecting \mathfrak{h} into \mathfrak{k}_j . We decompose this subspace orthogonally into

$$(\mathfrak{h})_{\mathfrak{k}_j} = (\mathfrak{h} \cap \mathfrak{k}_j) \oplus ((\mathfrak{h})_{\mathfrak{k}_j} \ominus (\mathfrak{h} \cap \mathfrak{k}_j)). \quad (5.3)$$

If we write $H_\mathfrak{k}^j = (H_\mathfrak{k}^j)_1 + (H_\mathfrak{k}^j)_2$ according to this decomposition, then $(H_\mathfrak{k}^j)_1 \in \mathfrak{h} \cap \mathfrak{k}_j \subset \mathfrak{h}$ and hence $(H_\mathfrak{k}^j)_2 + H^j \in \mathfrak{h}$. By this argument we may and do assume that

$$H_\mathfrak{k}^j \in (\mathfrak{h})_{\mathfrak{k}_j} \ominus (\mathfrak{h} \cap \mathfrak{k}_j).$$

In the next lemma we investigate the action of

$$f := \text{ad}(H_\mathfrak{k}^j).$$

Lemma 5.2. *For each $k = 0, 1, \dots, m_j$, we have*

- (1) f normalizes $(\nu_o W)_k$,
- (2) $f^2 = -c_k^2 \cdot \text{id}$ on $(\nu_o W)_k$.

Proof. We first show (1). Since $H_{\mathfrak{k}}^j \in \mathfrak{k}$, the map f is skewsymmetric. Therefore it is enough to show that f normalizes $\tau(\mathfrak{h}) \cap \mathfrak{b}_j$ and $\tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu$. Let $X \in \tau(\mathfrak{h}) \cap \mathfrak{b}_j$. There exists $X_{\mathfrak{k}} \in \mathfrak{k}_j$ such that $X_{\mathfrak{k}} + X \in \mathfrak{h}$. Since $[\mathfrak{a}_j, \mathfrak{m}_j] = 0$, we have

$$\mathfrak{h} \ni [H_{\mathfrak{k}}^j + H^j, X_{\mathfrak{k}} + X] = [H_{\mathfrak{k}}^j, X_{\mathfrak{k}}] + [H_{\mathfrak{k}}^j, X].$$

This concludes $f(X) = [H_{\mathfrak{k}}^j, X] \in \tau(\mathfrak{h}) \cap \mathfrak{b}_j$. Next, let $Y \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu$. There exists $Y_{\mathfrak{k}} \in \mathfrak{k}_j$ such that $Y_{\mathfrak{k}} + Y \in \mathfrak{h}$. By definition of H^j , we have

$$\mathfrak{h} \ni [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{k}} + Y] = [H_{\mathfrak{k}}^j, Y_{\mathfrak{k}}] + [H_{\mathfrak{k}}^j, Y] + \nu Y.$$

Hence we have $[H_{\mathfrak{k}}^j, Y] + \nu Y \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu$. Since $Y \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu$ by assumption, we conclude that $f(Y) = [H_{\mathfrak{k}}^j, Y] \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j^\nu$. This finishes (1).

To show (2), we need

$$[\mathfrak{h} \cap \mathfrak{k}_j, H_{\mathfrak{k}}^j] = 0. \quad (5.4)$$

Let $X \in \mathfrak{h} \cap \mathfrak{k}_j$. Then we have $[X, H_{\mathfrak{k}}^j + H^j] = [X, H_{\mathfrak{k}}^j] \in \mathfrak{h} \cap \mathfrak{k}_j$. On the other hand, since $\mathfrak{h} \cap \mathfrak{k}_j$ preserves the decomposition (5.3), we also have $[X, H_{\mathfrak{k}}^j] \in (\mathfrak{h})_{\mathfrak{k}_j} \ominus (\mathfrak{h} \cap \mathfrak{k}_j)$. This implies $[X, H_{\mathfrak{k}}^j] = 0$, which finishes the proof of (5.4).

We now prove (2). By Lemma 5.1 (4), each $(\nu_o W)_k$ is an irreducible $(\mathfrak{h} \cap \mathfrak{k}_j)$ -module. Hence (5.4) and Schur's Lemma yield that f is a multiple of the identity on the complexification of $(\nu_o W)_k$. Since all eigenvalues of f are purely imaginary, we conclude that $f^2 = -c_k^2 \cdot \text{id}$ on $(\nu_o W)_k$. q.e.d.

In the third step, we use the fact that \mathfrak{h} is a subalgebra, and prove that $\nu_o W$ is in the suitable position.

Proposition 5.3. *We have $\nu_o W \subset \mathfrak{b}_j \cong T_o B_j$ or $\nu_o W \subset \mathfrak{n}_j^1$.*

Proof. First we assume that $(\nu_o W)_1 = 0$, and show that $\nu_o W \subset \mathfrak{b}_j$. By assumption, we have $\mathfrak{n}_j^1 \subset \tau(\mathfrak{h})$. Then, for each $X, Y \in \mathfrak{n}_j^1$, there exist $X_{\mathfrak{k}}, Y_{\mathfrak{k}} \in \mathfrak{k}_j$ such that $X_{\mathfrak{k}} + X, Y_{\mathfrak{k}} + Y \in \mathfrak{h}$. Since \mathfrak{h} is a subalgebra, we have

$$\tau(\mathfrak{h}) \ni \tau([X_{\mathfrak{k}} + X, Y_{\mathfrak{k}} + Y]) = [X_{\mathfrak{k}}, Y] + [X, Y_{\mathfrak{k}}] + [X, Y].$$

Since \mathfrak{k}_j normalizes \mathfrak{n}_j^1 we get $[X_{\mathfrak{k}}, Y] + [X, Y_{\mathfrak{k}}] \in \mathfrak{n}_j^1 \subset \tau(\mathfrak{h})$ and therefore $[X, Y] \in \tau(\mathfrak{h})$. This means $\mathfrak{n}_j^2 \subset \tau(\mathfrak{h})$, since \mathfrak{n}_j^2 is generated by \mathfrak{n}_j^1 . Recall that \mathfrak{n}_j is generated by \mathfrak{n}_j^1 . Hence, using this argument inductively, we conclude that $\mathfrak{n}_j \subset \tau(\mathfrak{h})$. This finishes the first case.

We next assume that $(\nu_o W)_1 \neq 0$, which is the second case. We show that $\nu_o W \subset \mathfrak{n}_j^1$, that is,

$$(\nu_o W)_\nu = 0 \quad (\text{for } \nu \neq 1). \quad (5.5)$$

Assume that $(\nu_o W)_\nu \neq 0$ for some $\nu \neq 1$. Let $0 \neq X = X_0 + X_1 + \cdots + X_{m_j} \in \nu_o W$, where $X_k \in (\nu_o W)_k$. We have $X_1 \neq 0 \neq X_\nu$ by assumption and Lemma 5.1 (3). We put

$$Y_1 := \|X_\nu\|^2 X_1, \quad Y_\nu := -\|X_1\|^2 X_\nu.$$

Since $\langle X, Y_1 + Y_\nu \rangle = 0$, Lemma 5.1 (1) and the skewsymmetry of $\text{ad}(H)$ for all $H \in \mathfrak{h} \cap \mathfrak{k}_j$ imply

$$Y_1 + Y_\nu \in \tau(\mathfrak{h}). \quad (5.6)$$

There exists $Y_{\mathfrak{k}} \in \mathfrak{k}_j$ such that $Y_{\mathfrak{k}} + Y_1 + Y_{\nu} \in \mathfrak{h}$. This yields

$$\tau(\mathfrak{h}) \ni \tau([H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{k}} + Y_1 + Y_{\nu}]) = (f(Y_1) + Y_1) + (f(Y_{\nu}) + \nu Y_{\nu}). \quad (5.7)$$

By bracketing again we get

$$\begin{aligned} \tau(\mathfrak{h}) &\ni \tau([H_{\mathfrak{k}}^j + H^j, [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{k}} + Y_1 + Y_{\nu}]]) \\ &= (f^2(Y_1) + 2f(Y_1) + Y_1) + (f^2(Y_{\nu}) + 2\nu f(Y_{\nu}) + \nu^2 Y_{\nu}). \end{aligned}$$

From Lemma 5.2 we know that $f^2(Y_1) = -c_1^2 Y_1$ and $f^2(Y_{\nu}) = -c_{\nu}^2 Y_{\nu}$, and therefore

$$((1 - c_1^2)Y_1 + 2f(Y_1)) + ((\nu^2 - c_{\nu}^2)Y_{\nu} + 2\nu f(Y_{\nu})) \in \tau(\mathfrak{h}). \quad (5.8)$$

From (5.8) and (5.7) we get

$$(-1 - c_1^2)Y_1 + (\nu^2 - c_{\nu}^2 - 2\nu)Y_{\nu} + 2(\nu - 1)f(Y_{\nu}) \in \tau(\mathfrak{h}).$$

This and (5.6) yield

$$Y'_{\nu} := (\nu^2 - c_{\nu}^2 - 2\nu + 1 + c_1^2)Y_{\nu} + 2(\nu - 1)f(Y_{\nu}) \in \tau(\mathfrak{h}).$$

Therefore, $Y'_{\nu} \in \tau(\mathfrak{h}) \cap \mathfrak{b}_j$ if $\nu = 0$, and $Y'_{\nu} \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j'$ if $\nu \geq 2$. On the other hand, we have $Y_{\nu} \in (\nu_o W)_{\nu}$ by assumption, and $f(Y_{\nu}) \in (\nu_o W)_{\nu}$ by Lemma 5.2 (1). This means $Y'_{\nu} \in (\nu_o W)_{\nu}$, and we thus get $Y'_{\nu} = 0$. Since $\langle Y_{\nu}, f(Y_{\nu}) \rangle = 0$ and $\nu \neq 1$, we have $f(Y_{\nu}) = 0$. But this implies $c_{\nu} = 0$ and hence $\nu^2 - c_{\nu}^2 - 2\nu + 1 + c_1^2 > 0$, which contradicts $Y'_{\nu} = 0$. This finishes our claim (5.5) and shows that $\nu_o W \subset \mathfrak{n}_j^1$. q.e.d.

We next study the structure of \mathfrak{h} . Recall that there exists $H_{\mathfrak{k}}^j \in (\mathfrak{h})_{\mathfrak{k}_j} \ominus (\mathfrak{h} \cap \mathfrak{k}_j)$ such that $H_{\mathfrak{k}}^j + H^j \in \mathfrak{h}$. The next lemma shows that \mathfrak{h} fails to be compatible with the Langlands decomposition only for the abelian component \mathfrak{a}_j .

Lemma 5.4. *We have*

- (1) $\tau(\mathfrak{h}) \cap \mathfrak{n}_j = \mathfrak{h} \cap \mathfrak{n}_j$,
- (2) $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathbb{R}(H_{\mathfrak{k}}^j + H^j) \oplus (\mathfrak{h} \cap \mathfrak{n}_j)$.

Proof. Let S be a connected solvable subgroup of H which acts transitively on the singular orbit W (for the existence see e.g. Proposition 3.1 in [5]), and denote by \mathfrak{s} the Lie algebra of S . First of all, we show that

$$\tau(\mathfrak{s}) \cap \mathfrak{n}_j = \mathfrak{s} \cap \mathfrak{n}_j. \quad (5.9)$$

It is easy to see that $\mathfrak{s} \cap \mathfrak{n}_j \subset \tau(\mathfrak{s}) \cap \mathfrak{n}_j$. We now choose $Y_{\mathfrak{n}} \in (\tau(\mathfrak{s}) \cap \mathfrak{n}_j) \ominus (\mathfrak{s} \cap \mathfrak{n}_j)$, and show that $Y_{\mathfrak{n}} = 0$. Since $Y_{\mathfrak{n}} \in \tau(\mathfrak{s})$, there exists $Y_{\mathfrak{k}} \in \mathfrak{k}_j$ such that $Y_{\mathfrak{k}} + Y_{\mathfrak{n}} \in \mathfrak{s}$.

We now claim that

$$[H_{\mathfrak{k}}^j, Y_{\mathfrak{k}}] = 0. \quad (5.10)$$

To show (5.10) we define the solvable subalgebra $\mathfrak{s}' := \mathfrak{s} \cap (\mathfrak{k}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j)$. Let $\pi_{\mathfrak{k}} : \mathfrak{k}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j \rightarrow \mathfrak{k}_j$ be the canonical projection. Since $\mathfrak{k}_j \subset \mathfrak{m}_j$ normalizes $\mathfrak{a}_j \oplus \mathfrak{n}_j$, the map $\pi_{\mathfrak{k}}$ is a Lie algebra homomorphism, and therefore $\pi_{\mathfrak{k}}(\mathfrak{s}')$ is a solvable subalgebra of \mathfrak{k}_j . Since every solvable subalgebra of a compact Lie algebra is abelian, we conclude that $\pi_{\mathfrak{k}}(\mathfrak{s}')$ is an abelian subalgebra of \mathfrak{k}_j . By construction, we have $Y_{\mathfrak{k}} + Y_{\mathfrak{n}}, H_{\mathfrak{k}}^j + H^j \in \mathfrak{s}'$ and hence $H_{\mathfrak{k}}^j, Y_{\mathfrak{k}} \in \pi_{\mathfrak{k}}(\mathfrak{s}')$. This proves (5.10).

Our next claim is

$$[H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{n}}] \in \mathfrak{s} \cap \mathfrak{n}_j. \quad (5.11)$$

Recall that $H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{k}} + Y_{\mathfrak{n}} \in \mathfrak{s}$. Hence we have

$$\mathfrak{s} \ni [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{k}} + Y_{\mathfrak{n}}] = [H_{\mathfrak{k}}^j, Y_{\mathfrak{k}}] + [H^j, Y_{\mathfrak{k}}] + [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{n}}] = [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{n}}].$$

Note that the last equality follows from (5.10) and $[\mathfrak{k}_j, \mathfrak{a}_j] = 0$. As $H_{\mathfrak{k}}^j + H^j \in \mathfrak{k}_j \oplus \mathfrak{a}_j \subset \mathfrak{l}_j$ and \mathfrak{l}_j normalizes \mathfrak{n}_j , we also have $[H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{n}}] \in \mathfrak{n}_j$. Thus (5.11) has been proved.

Recall that $Y_{\mathfrak{n}} \in (\tau(\mathfrak{s}) \cap \mathfrak{n}_j) \ominus (\mathfrak{s} \cap \mathfrak{n}_j)$. From (5.11) and the skewsymmetry of $\text{ad}(H_{\mathfrak{k}}^j)$, we have

$$0 = \langle [H_{\mathfrak{k}}^j + H^j, Y_{\mathfrak{n}}], Y_{\mathfrak{n}} \rangle = \langle [H^j, Y_{\mathfrak{n}}], Y_{\mathfrak{n}} \rangle.$$

Recall that H^j determines the gradation $\mathfrak{n}_j = \bigoplus_{\nu=1}^{m_j} \mathfrak{n}_j^{\nu}$, and we therefore can write $Y_{\mathfrak{n}} = \sum_{\nu=1}^{m_j} Y_{\mathfrak{n}}^{\nu}$ with $Y_{\mathfrak{n}}^{\nu} \in \mathfrak{n}_j^{\nu}$. Hence we have

$$0 = \langle [H^j, Y_{\mathfrak{n}}], Y_{\mathfrak{n}} \rangle = \sum_{\nu=1}^{m_j} \nu \langle Y_{\mathfrak{n}}^{\nu}, Y_{\mathfrak{n}}^{\nu} \rangle,$$

which implies $Y_{\mathfrak{n}}^{\nu} = 0$ for all $\nu \in \{1, \dots, m_j\}$. We thus conclude that $Y_{\mathfrak{n}} = 0$, and therefore (5.9) has been proved.

We now prove statement (1) of the lemma. It is easy to see $\mathfrak{h} \cap \mathfrak{n}_j \subset \tau(\mathfrak{h}) \cap \mathfrak{n}_j$. To show the converse, note that $\tau(\mathfrak{h}) = \tau(\mathfrak{s})$ since $W = S \cdot o = H \cdot o$. Hence, by (5.9), we have

$$\tau(\mathfrak{h}) \cap \mathfrak{n}_j = \tau(\mathfrak{s}) \cap \mathfrak{n}_j = \mathfrak{s} \cap \mathfrak{n}_j \subset \mathfrak{s} \subset \mathfrak{h}.$$

This proves $\tau(\mathfrak{h}) \cap \mathfrak{n}_j \subset \mathfrak{h} \cap \mathfrak{n}_j$ and hence (1) holds.

We now prove statement (2) of the lemma. It is easy to see “ \supset ” of (2). To show “ \subset ”, we choose $X \in \mathfrak{h}$ and write

$$X = X_{\mathfrak{m}} + X_{\mathfrak{a}} + X_{\mathfrak{n}}$$

according to the Langlands decomposition $\mathfrak{h} \subset \mathfrak{q}_j = \mathfrak{m}_j \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j$. By definition of \mathfrak{a}_j we can write $X_{\mathfrak{a}} = cH^j$ with some $c \in \mathbb{R}$, and hence we can write X as

$$X = (X_{\mathfrak{m}} - cH_{\mathfrak{k}}^j) + c(H_{\mathfrak{k}}^j + H^j) + X_{\mathfrak{n}}.$$

One can easily see that

$$X_{\mathfrak{m}} - cH_{\mathfrak{k}}^j \in \mathfrak{m}_j, \quad c(H_{\mathfrak{k}}^j + H^j) \in \mathbb{R}(H_{\mathfrak{k}}^j + H^j), \quad X_{\mathfrak{n}} \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j.$$

Note that $X_{\mathfrak{n}} \in \tau(\mathfrak{h}) \cap \mathfrak{n}_j = \mathfrak{h} \cap \mathfrak{n}_j$ from (1). Hence we have

$$\mathfrak{h} \ni X - c(H_{\mathfrak{k}}^j + H^j) - X_{\mathfrak{n}} = X_{\mathfrak{m}} - cH_{\mathfrak{k}}^j.$$

This yields $X_{\mathfrak{m}} - cH_{\mathfrak{k}}^j \in \mathfrak{h} \cap \mathfrak{m}_j$ and proves $X \in (\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathbb{R}(H_{\mathfrak{k}}^j + H^j) \oplus (\mathfrak{h} \cap \mathfrak{n}_j)$. This finishes the proof of (2). q.e.d.

We next show that \mathfrak{h} can be replaced by a simpler subalgebra with an orbit equivalent action.

Lemma 5.5. *The action of H is orbit equivalent to the action of the connected Lie subgroup H' of Q_j with Lie algebra $\mathfrak{h}' := (\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathfrak{a}_j \oplus (\mathfrak{h} \cap \mathfrak{n}_j)$.*

Proof. From Proposition 5.3 we know that $\nu_o W \subset \mathfrak{b}_j$ or $\nu_o W \subset \mathfrak{n}_j^1$, and thus we have

$$\mathfrak{h} \cap \mathfrak{n}_j = \mathfrak{n}_j \quad \text{or} \quad \mathfrak{h} \cap \mathfrak{n}_j = \mathfrak{n}_j \ominus \nu_o W = (\mathfrak{n}_j^1 \ominus \nu_o W) \oplus \left(\bigoplus_{\nu > 1} \mathfrak{n}_j^{\nu} \right). \quad (5.12)$$

First of all, we show that \mathfrak{h}' is a subalgebra. It is easy to see that $(\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathfrak{a}_j$ is a subalgebra, and from (5.12) we see that $\mathfrak{a}_j \oplus (\mathfrak{h} \cap \mathfrak{n}_j)$ is a subalgebra. Since \mathfrak{m}_j normalizes \mathfrak{n}_j , we also have $[\mathfrak{h} \cap \mathfrak{m}_j, \mathfrak{h} \cap \mathfrak{n}_j] \subset \mathfrak{h} \cap \mathfrak{n}_j$. Altogether this implies that \mathfrak{h}' is a subalgebra.

Next we prove that $\mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}'$ is a subalgebra. Since \mathfrak{h}' is a subalgebra, it is enough to show that

$$(i) [H_\mathfrak{k}^j, \mathfrak{h} \cap \mathfrak{m}_j] \subset \mathfrak{h} \cap \mathfrak{m}_j, \quad (ii) [H_\mathfrak{k}^j, \mathfrak{a}_j] = 0, \quad (iii) [H_\mathfrak{k}^j, \mathfrak{h} \cap \mathfrak{n}_j] \subset \mathfrak{h} \cap \mathfrak{n}_j.$$

Let $X \in \mathfrak{h} \cap \mathfrak{m}_j$. Since $H_\mathfrak{k}^j \in \mathfrak{k}_j \subset \mathfrak{m}_j$ and \mathfrak{m}_j is a subalgebra, we have $[H_\mathfrak{k}^j, X] \in \mathfrak{m}_j$. Furthermore, since H^j centralizes \mathfrak{m}_j , we have $[H_\mathfrak{k}^j, X] = [H_\mathfrak{k}^j + H^j, X] \in \mathfrak{h}$. Altogether this gives $[H_\mathfrak{k}^j, X] \in \mathfrak{h} \cap \mathfrak{m}_j$, which implies (i). The claim (ii) is easy to verify. If $\mathfrak{h} \cap \mathfrak{n}_j = \mathfrak{n}_j$, then it is easy to see that (iii) holds. If $\mathfrak{h} \cap \mathfrak{n}_j = \mathfrak{n}_j \ominus \nu_o W$, then $\text{ad}(H_\mathfrak{k}^j)$ normalizes $\nu_o W$ by Lemma 5.2, and hence normalizes $\mathfrak{n}_j \ominus \nu_o W$ by skewsymmetry of $\text{ad}(H_\mathfrak{k}^j)$. This finishes the proof of (iii).

We now consider the three subalgebras \mathfrak{h} , $\mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}'$ and \mathfrak{h}' . By construction, we have $\mathfrak{h}, \mathfrak{h}' \subset \mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}'$. Denote by H'' the connected Lie subgroup of Q_j with Lie algebra $\mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}'$. We will now prove that the actions of H , H' and H'' are orbit equivalent to each other.

We first show that the actions of H and H'' are orbit equivalent. Since $H \subset H''$, we obviously have $W = H \cdot o \subset H'' \cdot o$. However, since $T_o W = \tau(\mathfrak{h}) = \tau(\mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}')$ and both orbits are connected and complete, we conclude that $W = H \cdot o = H'' \cdot o$. By assumption the action of H is of cohomogeneity one, and therefore the action of H'' must be of cohomogeneity one as well. This implies that the actions of H and H'' are orbit equivalent.

We next show that the actions of H' and H'' are orbit equivalent. Since $H' \subset H''$ and $\tau(\mathfrak{h}') = \tau(\mathbb{R}H_\mathfrak{k}^j \oplus \mathfrak{h}') = T_o W$, we conclude that $H' \cdot o = H'' \cdot o = W$. By construction, we have $\mathfrak{h}' \cap \mathfrak{k}_j = \mathfrak{h} \cap \mathfrak{k}_j$, which implies that the slice representations of H' and H at o are the same. Since the action of H is of cohomogeneity one by assumption, the action of H' is of cohomogeneity one as well. Thus, since both actions are of cohomogeneity one and have the same singular orbit, we conclude that these actions are orbit equivalent.

We thus have proved that the actions of H and H' are orbit equivalent. q.e.d.

According to Proposition 5.3, the normal space of the singular orbit is either tangent to the (totally geodesic) semisimple part or to the nilpotent part of the horospherical decomposition of M induced by Φ_j . We now distinguish these two cases.

Proposition 5.6. *If $\nu_o W \subset \mathfrak{b}_j$, then the action of H on M is orbit equivalent to the canonical extension of a cohomogeneity one action on the boundary component B_j of M .*

Proof. Assume that $\nu_o W \subset \mathfrak{b}_j$. According to Lemma 5.5 we can assume that

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j.$$

Note that \mathfrak{m}_j is reductive and we have the Lie algebra direct sum decomposition $\mathfrak{m}_j = \mathfrak{z}_j \oplus \mathfrak{g}_j$, where \mathfrak{z}_j is the center of \mathfrak{m}_j . Therefore the canonical projection $\pi_\mathfrak{g} : \mathfrak{m}_j \rightarrow \mathfrak{g}_j$ with respect to this decomposition is a Lie algebra homomorphism and $\mathfrak{h}' := \pi_\mathfrak{g}(\mathfrak{h} \cap \mathfrak{m}_j)$ is a subalgebra of \mathfrak{g}_j . Let H' be the connected subgroup of G_j with Lie algebra \mathfrak{h}' . We claim that H' acts on $B_j = M_j/K_j = G_j/(G_j \cap K_j)$ with cohomogeneity one and the canonical extension of this action to M is orbit equivalent to the action of H on M .

We first prove that H' acts on B_j with cohomogeneity one. For simplicity we will identify the subalgebras and the corresponding connected Lie subgroups. At first we

consider the action of $\mathfrak{h} \cap \mathfrak{m}_j$ on B_j . The slice representation of this action is the action of $\mathfrak{h} \cap \mathfrak{k}_j$ on $\nu_o W$, which coincides with the slice representation of the action of H on M . Therefore, $\mathfrak{h} \cap \mathfrak{k}_j$ acts transitively on the unit sphere in $\nu_o W$, and hence the action of $\mathfrak{h} \cap \mathfrak{m}_j$ on B_j is of cohomogeneity one. Next we consider

$$\mathfrak{h} \cap \mathfrak{m}_j \subset \mathfrak{h}' \oplus \mathfrak{z}_j.$$

Since $\tau(\mathfrak{h} \cap \mathfrak{m}_j) = \tau(\mathfrak{h}' \oplus \mathfrak{z}_j)$, the action of $\mathfrak{h}' \oplus \mathfrak{z}_j$ on B_j is also of cohomogeneity one. Finally we consider

$$\mathfrak{h}' \oplus \mathfrak{z}_j \supset \mathfrak{h}'.$$

Since $\tau(\mathfrak{h} \cap \mathfrak{m}_j) = \tau(\mathfrak{h}')$, their orbits through o coincide. Furthermore, since \mathfrak{z}_j acts trivially on $\nu_o W$, the slice representations of these actions are equivalent. Therefore we conclude that the action of H' on B_j is of cohomogeneity one.

We now consider the canonical extension H_j^Λ of H' to M . By definition, we have

$$\mathfrak{h}_j^\Lambda = \mathfrak{h}' \oplus \mathfrak{a}_j \oplus \mathfrak{n}_j.$$

By a similar argument as above, one can show that the following three actions are orbit equivalent:

$$\mathfrak{h}_j^\Lambda \subset \mathfrak{h}_j^\Lambda \oplus \mathfrak{z}_j \supset \mathfrak{h}.$$

Therefore, the action of H is orbit equivalent to the action of the canonical extension H_j^Λ of H' . q.e.d.

We now turn our attention to cohomogeneity one actions with $\nu_o W \subset \mathfrak{n}_j^1$.

Proposition 5.7. *If $\nu_o W \subset \mathfrak{n}_j^1$, then the action of H on M is orbit equivalent to the action of $H_{j,\mathfrak{v}}$ for some subspace $\mathfrak{v} \subset \mathfrak{n}_j^1$.*

Proof. Assume that $\nu_o W \subset \mathfrak{n}_j^1$. By means of Lemma 5.5 we can assume that

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathfrak{a}_j \oplus (\mathfrak{n}_j \ominus \nu_o W).$$

Let $\mathfrak{v} := \nu_o W$ and recall that

$$\mathfrak{h}_{j,\mathfrak{v}} = N_{\mathfrak{l}_j}(\mathfrak{n}_{j,\mathfrak{v}}) \oplus \mathfrak{n}_{j,\mathfrak{v}} = N_{\mathfrak{l}_j}(\mathfrak{n}_{j,\mathfrak{v}}) \oplus (\mathfrak{n}_j \ominus \nu_o W).$$

We have $\mathfrak{h} \subset \mathfrak{h}_{j,\mathfrak{v}}$ since \mathfrak{h} is a subalgebra and hence $(\mathfrak{h} \cap \mathfrak{m}_j) \oplus \mathfrak{a}_j$ normalizes $\mathfrak{n}_{j,\mathfrak{v}}$. One can also see that $\tau(\mathfrak{h}) = \tau(\mathfrak{h}_{j,\mathfrak{v}})$, and therefore $H \cdot o = H_{j,\mathfrak{v}} \cdot o$. Since the actions of H and $H_{j,\mathfrak{v}}$ are of cohomogeneity one, these actions are orbit equivalent. q.e.d.

From the previous two propositions we obtain the main result of this section.

Theorem 5.8. *Let M be a connected Riemannian symmetric space of noncompact type, and let H be a connected subgroup of G which acts on M with cohomogeneity one and has a singular orbit. Assume that H is contained in a proper maximal parabolic subgroup Q_j of G . Then the action of H on M is orbit equivalent to*

- (i) *a cohomogeneity one action on M obtained by canonical extension of a cohomogeneity one action on the maximal boundary component B_j , or*
- (ii) *a cohomogeneity one action on M given by $H_{j,\mathfrak{v}}$ for some subspace $\mathfrak{v} \subset \mathfrak{n}_j^1$.*

We emphasize that in Theorem 5.8 the symmetric space M can be reducible. As a consequence of this result we also see that a non-totally geodesic singular orbit of a cohomogeneity one action on M which is not a canonical extension contains a maximal boundary component of M .

This finishes the proof of Theorem 1.1.

6. SOME EXPLICIT CLASSIFICATIONS

In this section we present explicit classifications of cohomogeneity one actions (up to orbit equivalence) for some symmetric spaces of rank two. We have chosen symmetric spaces for which the Lie algebra \mathfrak{g} of the isometry group is a split real form of its complexification $\mathfrak{g}^{\mathbb{C}}$. In order to describe these we recall briefly the classification of cohomogeneity one actions on a real hyperbolic space $\mathbb{R}H^n$ (see [5] for further details).

Theorem 6.1. *Every cohomogeneity one action on the real hyperbolic space $\mathbb{R}H^n = SO_{1,n}^o/SO_n$ is orbit equivalent to one of the following actions:*

- (1) *The action of $SO_{1,k}^o \times SO_{n-k} \subset SO_{1,n}^o$ for some $k \in \{0, \dots, n-1\}$. For $k < n-1$ the action has exactly one singular orbit, namely a totally geodesic $\mathbb{R}H^k \subset \mathbb{R}H^n$. For $k = n-1$ the orbits form a foliation, and one of the orbits is a totally geodesic $\mathbb{R}H^{n-1}$.*
- (ii) *The action of the nilpotent subgroup N in an Iwasawa decomposition $SO_{1,n}^o = SO_n AN$. The orbits form a foliation on $\mathbb{R}H^n$ by horospheres.*

Since $SO_{1,n}^o$ acts transitively on $\mathbb{R}H^n$ and the isotropy group at a point is isomorphic to SO_n , it can easily be seen that orbit equivalence can always be achieved by an isometry in $SO_{1,n}^o$.

6.1. The symmetric space $M = SL_3(\mathbb{R})/SO_3$. The symmetric space $M = SL_3(\mathbb{R})/SO_3$ has rank 2 and dimension 5. The root system is of type (A_2) and all multiplicities are equal to 1. The positive roots are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ and the nilpotent subalgebra \mathfrak{n} of $\mathfrak{g} = \mathfrak{sl}_3(\mathbb{R})$ is given by

$$\mathfrak{n} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}.$$

The maximal abelian subalgebra \mathfrak{a} has dimension 2 and is spanned by the two root vectors H_{α_1} and H_{α_2} . The Chevalley decomposition $\mathfrak{q}_2 = \mathfrak{l}_2 \oplus \mathfrak{n}_2$ of the parabolic subalgebra \mathfrak{q}_2 corresponding to $\Phi_2 = \{\alpha_1\}$ is given by

$$\mathfrak{l}_2 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R} \quad \text{and} \quad \mathfrak{n}_2 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}.$$

The orbit $F_2 = L_2 \cdot o$ is isometric to the Riemannian product $\mathbb{R}H^2 \times \mathbb{E}$, and the corresponding boundary component B_2 is the real hyperbolic plane $\mathbb{R}H^2$.

Theorem 6.2. *Each cohomogeneity one action on $M = SL_3(\mathbb{R})/SO_3$ is orbit equivalent to one of the following cohomogeneity one actions on M :*

- (1) *The action of the subgroup H_ℓ of $SL_3(\mathbb{R})$ with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ is a one-dimensional linear subspace of \mathfrak{a} . The orbits form a Riemannian foliation on M and all orbits are isometrically congruent to each other.

- (2) *The action of the subgroup H_1 of $SL_3(\mathbb{R})$ with Lie algebra*

$$\mathfrak{h}_1 = (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{g}_{\alpha_1}.$$

The orbits form a Riemannian foliation on M and there is exactly one minimal orbit $H_1 \cdot o$.

- (3) *The action of $SL_2(\mathbb{R}) \times \mathbb{R}^+ \subset SL_3(\mathbb{R})$ with Lie algebra*

$$\mathfrak{l}_2 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}.$$

This action has a totally geodesic singular orbit isometric to $\mathbb{R}H^2 \times \mathbb{E}$.

- (4) *The action of the connected subgroup H of $SL_3(\mathbb{R})$ with Lie algebra*

$$\mathfrak{h} = \mathfrak{k}_{\alpha_1} \oplus (\mathfrak{a} \ominus \mathbb{R}H_{\alpha_1}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_1}),$$

where $\mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2$ is the Lie algebra of the isotropy group of the isometry group of the boundary component $B_2 = \mathbb{R}H^2$. This action has a minimal $\mathbb{R}H^3 \subset M$ as a singular orbit and can be constructed by canonical extension of the cohomogeneity one action on $B_2 = \mathbb{R}H^2$ of M which has a single point as an orbit.

Proof. For the classification we have to consider the different cases in Theorem 1.1. If the orbits form a Riemannian foliation, we obtain the actions described in (1) and (2). The action described in (3) is the only one corresponding to case (2)(i) in Theorem 1.1 according to [4]. We now consider an action as described in Theorem 1.1 (2)(ii). The symmetric space M has, up to isometric congruence, only one boundary component with rank one, namely the real hyperbolic plane $B_2 = \mathbb{R}H^2$. There is, up to orbit equivalence, exactly one cohomogeneity one action on $\mathbb{R}H^2$ with a singular orbit, namely the action on $\mathbb{R}H^2$ by the isotropy group $K_{\alpha_1} \cong SO_2$. The canonical extension of this action leads to the action described in (4). It remains to investigate case (b) in (2)(ii) of Theorem 1.1. For $\Phi_2 = \{\alpha_1\}$ we have $\mathfrak{n}_2^1 = \mathfrak{n}_2 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$, and therefore $\mathfrak{v} = \mathfrak{n}_2^1$ for dimension reasons. It is easy to see that $N_{K_2}^o(\mathfrak{v}) = K_2 \cong SO_2$ acts transitively on the unit sphere in \mathfrak{v} . Moreover, the normalizer of $\mathfrak{n}_2 \ominus \mathfrak{v} = \{0\}$ in L_2 is clearly L_2 , which acts transitively on $F_2 = \mathbb{R}H^2 \times \mathbb{E}$. The construction method in (2)(ii)(b) therefore leads to the cohomogeneity one action on M by L_2 , which is the action described in (3). The case $\Phi_1 = \{\alpha_2\}$ does not lead to anything new because of the Dynkin diagram symmetry. q.e.d.

6.2. The symmetric space $SO_{2,3}^o/SO_2SO_3 = G_2^*(\mathbb{R}^5)$. The symmetric space $M = SO_{2,3}^o/SO_2SO_3$ has rank 2 and dimension 6. The root system is of type (B_2) , the Dynkin diagram is

$$\begin{array}{ccc} \circ & \rightleftarrows & \circ \\ \alpha_1 & & \alpha_2 \end{array}$$

and all multiplicities are equal to 1. The maximal abelian subalgebra \mathfrak{a} has dimension 2 and is spanned by the two root vectors H_{α_1} and H_{α_2} . Both boundary components B_1 and B_2 are isometric to a real hyperbolic plane $\mathbb{R}H^2$. However B_1 and B_2 are not isometrically congruent to each other.

Theorem 6.3. *Each cohomogeneity one action on $M = SO_{2,3}^o/SO_2SO_3$ is orbit equivalent to one of the following cohomogeneity one actions on M :*

- (1) *The action of the subgroup H_ℓ of $SO_{2,3}^o$ with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ is a one-dimensional linear subspace of \mathfrak{a} . The orbits form a Riemannian foliation on M and all orbits are isometrically congruent to each other.

- (2) *The action of the subgroup H_i , $i \in \{1, 2\}$, of $SO_{2,3}^o$ with Lie algebra*

$$\mathfrak{h}_i = (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{g}_{\alpha_i}.$$

The orbits form a Riemannian foliation on M and there is exactly one minimal orbit $H_i \cdot o$.

- (3) *The action of $SO_{1,3}^o \subset SO_{2,3}^o$. This action has a totally geodesic singular orbit isometric to the real hyperbolic space $\mathbb{R}H^3$.*

- (4) The action of $SO_{2,2}^o \subset SO_{2,3}^o$. This action has a totally geodesic singular orbit isometric to the Riemannian product $\mathbb{R}H^2 \times \mathbb{R}H^2$.
- (5) The action of the subgroup H_1^Λ of $SO_{2,3}^o$ with Lie algebra

$$\mathfrak{h}_1^\Lambda = \mathfrak{k}_{\alpha_2} \oplus (\mathfrak{a} \ominus \mathbb{R}H_{\alpha_2}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_2}),$$

where $\mathfrak{k}_{\alpha_2} \cong \mathfrak{so}_2$ is the Lie algebra of the isotropy group of the isometry group of the boundary component $B_1 \cong \mathbb{R}H^2$. The action of H_1^Λ has a minimal real hyperbolic space $\mathbb{R}H^4 \subset M$ as a singular orbit and can be constructed by canonical extension of the cohomogeneity one action on the boundary component $B_1 = \mathbb{R}H^2$ which has a single point as an orbit.

- (6) The action of the subgroup H_2^Λ of $SO_{2,3}^o$ with Lie algebra

$$\mathfrak{h}_2^\Lambda = \mathfrak{k}_{\alpha_1} \oplus (\mathfrak{a} \ominus \mathbb{R}H_{\alpha_1}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_1}),$$

where $\mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2$ is the Lie algebra of the isotropy group of the isometry group of the boundary component $B_2 \cong \mathbb{R}H^2$. The action of H_2^Λ has a minimal complex hyperbolic plane $\mathbb{C}H^2 \subset M$ as a singular orbit and can be constructed by canonical extension of the cohomogeneity one action on the boundary component $B_2 = \mathbb{R}H^2$ which has a single point as an orbit.

Proof. For the classification we have to consider the different cases in Theorem 1.1. If the orbits form a Riemannian foliation, we obtain the actions in (1) and (2). The actions in (3) and (4) are the only ones corresponding to case (2)(i) in Theorem 1.1 according to [4]. We now consider an action as described in Theorem 1.1 (2)(ii). The symmetric space M has two maximal boundary components B_1 and B_2 . Both B_1 and B_2 are isometric to $\mathbb{R}H^2$ with a suitable constant curvature metric, but they are not isometrically congruent in M . There is, up to orbit equivalence, exactly one cohomogeneity one action on $\mathbb{R}H^2$ with a singular orbit, namely the action on $\mathbb{R}H^2$ by the isotropy group SO_2 . The canonical extension of this action leads to the actions in (5) and (6). It remains to investigate case (b) in (2)(ii) of Theorem 1.1. We have to consider two possible choices of subsystems of $\Lambda = \{\alpha_1, \alpha_2\}$, namely $\Phi_1 = \{\alpha_2\}$ and $\Phi_2 = \{\alpha_1\}$.

In case of Φ_1 we have

$$\begin{aligned} \mathfrak{n}_1^1 &= \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R}^3, \\ \mathfrak{l}_1 &= \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}, \\ \mathfrak{k}_1 &= \mathfrak{k}_{\alpha_2} \cong \mathfrak{so}_2. \end{aligned}$$

Since \mathfrak{k}_1 is one-dimensional, \mathfrak{v} must be a 2-dimensional linear subspace of \mathfrak{n}_1^1 . Let \mathfrak{v} be a \mathfrak{k}_1 -invariant 2-dimensional subspace of \mathfrak{n}_1^1 . In order to get a cohomogeneity one action, the normalizer $N_{L_1}(\mathfrak{n}_1^1 \ominus \mathfrak{v})$ must act transitively on $F_1 = L_1 \cdot o \cong \mathbb{R}H^2 \times \mathbb{E}$. The only subgroups of $SL_2(\mathbb{R})$ acting transitively on $\mathbb{R}H^2$ are $SL_2(\mathbb{R})$ itself and the parabolic subgroups of $SL_2(\mathbb{R})$. However, $\mathfrak{m}_1 \cong \mathfrak{sl}_2(\mathbb{R})$ acts irreducibly on \mathfrak{n}_1^1 , and hence $N_{L_2}(\mathfrak{n}_1^1 \ominus \mathfrak{v})$ cannot be equal to $SL_2(\mathbb{R})$. Since K_1 is compact and normalizes \mathfrak{v} , it also normalizes $\mathfrak{n}_1^1 \ominus \mathfrak{v}$. If a parabolic subgroup of $SL_2(\mathbb{R})$ would normalize $\mathfrak{n}_1^1 \ominus \mathfrak{v}$, then the entire group $SL_2(\mathbb{R})$ would normalize $\mathfrak{n}_1^1 \ominus \mathfrak{v}$, which cannot happen. We thus conclude that $N_{L_1}(\mathfrak{n}_1^1 \ominus \mathfrak{v})$ cannot act transitively on F_1 . This implies that there is no cohomogeneity one action on M which can be constructed from the choice of Φ_1 .

In case of Φ_2 we have

$$\begin{aligned} \mathfrak{n}_2^1 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2, \\ \mathfrak{n}_2^2 &= \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathbb{R}, \\ \mathfrak{l}_2 &= \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R}, \\ \mathfrak{k}_2 &= \mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_1. \end{aligned}$$

The only possible choice for \mathfrak{v} is therefore $\mathfrak{v} = \mathfrak{n}_2^1$. The normalizer $N_{L_2}(\mathfrak{n}_2^1 \ominus \mathfrak{v})$ is of course L_2 , and therefore we get a cohomogeneity one action on M . However, this action has $(L_2 N_2^2) \cdot o \cong \mathbb{R}H^2 \times \mathbb{R}H^2$ as a totally geodesic singular orbit, which we already listed in (4). q.e.d.

6.3. The symmetric space G_2^2/SO_4 . The symmetric space $M = G_2^2/SO_4$ has rank 2 and dimension 8. The root system is of type (G_2) , the Dynkin diagram is

$$\begin{array}{c} \alpha_1 \quad \alpha_2 \\ \circ \longleftarrow \circ \end{array},$$

and all multiplicities are equal to 1. The maximal abelian subalgebra \mathfrak{a} has dimension two and is spanned by the two root vectors H_{α_1} and H_{α_2} . Both boundary components B_1 and B_2 are isometric to a real hyperbolic plane $\mathbb{R}H^2$. However, B_1 and B_2 are not isometrically congruent to each other.

Theorem 6.4. *Each cohomogeneity one action on $M = G_2^2/SO_4$ is orbit equivalent to one of the following cohomogeneity one actions on M :*

- (1) *The action of the subgroup H_ℓ of G_2^2 with Lie algebra*

$$\mathfrak{h}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n},$$

where ℓ is a one-dimensional linear subspace of \mathfrak{a} . The orbits form a Riemannian foliation on M and all orbits are isometrically congruent to each other.

- (2) *The action of the subgroup H_i , $i \in \{1, 2\}$, of G_2^2 with Lie algebra*

$$\mathfrak{h}_i = (\mathfrak{a} \oplus \mathfrak{n}) \ominus \mathfrak{g}_{\alpha_i}.$$

The orbits form a Riemannian foliation on M and there is exactly one minimal orbit $H_i \cdot o$.

- (3) *The action of $SU_{1,2} \subset G_2^2$. This action has a totally geodesic singular orbit isometric to the complex hyperbolic plane $\mathbb{C}H^2$.*
(4) *The action of $SL_3(\mathbb{R}) \subset G_2^2$. This action has a totally geodesic singular orbit isometric to the symmetric space $SL_3(\mathbb{R})/SO_3$.*
(5) *The action of the subgroup H_1^Λ of G_2^2 with Lie algebra*

$$\mathfrak{h}_1^\Lambda = \mathfrak{k}_{\alpha_2} \oplus (\mathfrak{a} \ominus \mathbb{R}H_{\alpha_2}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_2}),$$

where $\mathfrak{k}_{\alpha_2} \cong \mathfrak{so}_2$ is the Lie algebra of the isotropy group of the isometry group of the boundary component $B_1 \cong \mathbb{R}H^2$. This action has a 6-dimensional minimal singular orbit and can be constructed by canonical extension of the cohomogeneity one action on the boundary component $B_1 = \mathbb{R}H^2$ which has a single point as an orbit.

- (6) *The action of the subgroup H_2^Λ of $SO_{2,3}^o$ with Lie algebra*

$$\mathfrak{h}_2^\Lambda = \mathfrak{k}_{\alpha_1} \oplus (\mathfrak{a} \ominus \mathbb{R}H_{\alpha_1}) \oplus (\mathfrak{n} \ominus \mathfrak{g}_{\alpha_1}),$$

where $\mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2$ is the Lie algebra of the isotropy group of the isometry group of the boundary component $B_2 \cong \mathbb{R}H^2$. This action has a minimal complex hyperbolic space $\mathbb{C}H^3 \subset M$ as a singular orbit and can be constructed by canonical extension of the cohomogeneity one action on the boundary component $B_2 = \mathbb{R}H^2$ which has a single point as an orbit.

(7) The action of the subgroup $H_{1,\mathfrak{v}}$ of G_2^2 with $\mathfrak{v} = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}$ and Lie algebra

$$\mathfrak{h}_{1,\mathfrak{v}} = \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2}.$$

This action has a 6-dimensional minimal singular orbit.

Proof. The argumentation for cases (1) to (6) is analogous to the one given in the proof of Theorem 6.3. We now consider the two possible choices of subsystems of $\Lambda = \{\alpha_1, \alpha_2\}$, namely $\Phi_1 = \{\alpha_2\}$ and $\Phi_2 = \{\alpha_1\}$.

In case of Φ_1 we have $\mathfrak{n}_1^1 = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \cong \mathbb{R}^2$. The only possible choice for \mathfrak{v} is therefore $\mathfrak{v} = \mathfrak{n}_1^1$. This was discussed in detail in subsection 4.2, where we showed that this leads to the cohomogeneity one action described in (7). This action cannot be orbit equivalent to the one in (5) or (6), as it contains a maximal flat of M , whereas the two singular orbits in (5) and (6) do not contain a maximal flat of M .

Finally, we consider Φ_2 . In this case we have

$$\begin{aligned} \mathfrak{n}_2^1 &= \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{2\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+\alpha_2} \cong \mathbb{R}^4 \\ \mathfrak{n}_2^2 &= \mathfrak{g}_{3\alpha_1+2\alpha_2} \\ \mathfrak{l}_2 &= \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{sl}_2(\mathbb{R}) \oplus \mathbb{R} \\ \mathfrak{k}_2 &= \mathfrak{k}_{\alpha_1} \cong \mathfrak{so}_2. \end{aligned}$$

Since \mathfrak{k}_2 is one-dimensional, \mathfrak{v} must be a 2-dimensional linear subspace of \mathfrak{n}_1^1 . Let \mathfrak{v} be a \mathfrak{k}_2 -invariant subspace of \mathfrak{n}_2^1 . In order to get a cohomogeneity one action, the normalizer $N_{L_2}(\mathfrak{n}_2^1 \ominus \mathfrak{v})$ must act transitively on $F_2 = L_2 \cdot o \cong \mathbb{R}H^2 \times \mathbb{E}$. The only subgroups of $SL_2(\mathbb{R})$ acting transitively on $\mathbb{R}H^2$ are $SL_2(\mathbb{R})$ itself and the parabolic subgroups of $SL_2(\mathbb{R})$. However, $\mathfrak{m}_2 \cong \mathfrak{sl}_2(\mathbb{R})$ acts irreducibly on \mathfrak{n}_2^1 , and hence $N_{L_2}(\mathfrak{n}_2^1 \ominus \mathfrak{v})$ cannot be equal to $SL_2(\mathbb{R})$. Since K_2 is compact and normalizes \mathfrak{v} , it also normalizes $\mathfrak{n}_2^1 \ominus \mathfrak{v}$. If a parabolic subgroup of $SL_2(\mathbb{R})$ would normalize $\mathfrak{n}_2^1 \ominus \mathfrak{v}$, then the entire group $SL_2(\mathbb{R})$ would normalize $\mathfrak{n}_2^1 \ominus \mathfrak{v}$, which cannot happen. We thus conclude that $N_{L_2}(\mathfrak{n}_2^1 \ominus \mathfrak{v})$ cannot act transitively on F_2 . This implies that there is no cohomogeneity one action on M which can be constructed from the choice of Φ_2 . q.e.d.

REFERENCES

- [1] J. Berndt and M. Brück, Cohomogeneity one actions on hyperbolic spaces, *J. Reine Angew. Math.* **541** (2001), 209–235.
- [2] J. Berndt, S. Console and C. Olmos, *Submanifolds and holonomy*, Chapman & Hall/CRC, Boca Raton, 2003.
- [3] J. Berndt and H. Tamaru, Homogeneous codimension one foliations on noncompact symmetric spaces, *J. Differential Geom.* **63** (2003), 1–40.
- [4] J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit, *Tôhoku Math. J.* **56** (2004), 163–177.
- [5] J. Berndt and H. Tamaru, Cohomogeneity one actions on noncompact symmetric spaces of rank one, *Trans. Amer. Math. Soc.* **359** (2007), 3425–3438.
- [6] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Birkhäuser, Boston, 2006.

- [7] É. Cartan, Familles de surfaces isoparamétriques dans les espaces à courbure constante, *Ann. di Mat.* **17** (1938), 177–191.
- [8] A.J. Di Scala and C. Olmos, A geometric proof of the Karpelevich-Mostow theorem, *Bull. London Math. Soc.* **41** (2009), 634–638.
- [9] P.B. Eberlein, *Geometry of nonpositively curved manifolds*, University of Chicago Press, Chicago, London, 1996.
- [10] S. Helgason, *Geometric analysis on symmetric spaces*, American Mathematical Society, Providence, RI, 1994.
- [11] W.Y. Hsiang and H.B. Lawson Jr., Minimal submanifolds of low cohomogeneity, *J. Differential Geom.* **5** (1971), 1–38.
- [12] N. Iwahori, On discrete reflection groups on symmetric Riemannian manifolds, in: *Proc. U.S.-Japan Seminar on Differential Geometry (Kyoto, 1965)*, Nippon Hyoronsha, Tokyo, 1966, 57–62.
- [13] K. Iwata, Classification of compact transformation groups on cohomology quaternion projective spaces with codimension one orbits, *Osaka J. Math.* **15** (1978), 475–508.
- [14] K. Iwata, Compact transformation groups on rational cohomology Cayley projective planes, *Tôhoku Math. J. (2)* **33** (1981), 429–442.
- [15] S. Kaneyuki and H. Asano, Graded Lie algebras and generalized Jordan triple systems, *Nagoya Math. J.* **112** (1988), 81–115.
- [16] F.I. Karpelevic, Surfaces of transitivity of a semisimple subgroup of the group of motions of a symmetric space (in Russian), *Dokl. Akad. Nauk SSSR (N.S.)* **93** (1953), 401–404.
- [17] A.W. Knap, *Lie groups beyond an introduction, second edition*, Birkhäuser, Boston, 2005.
- [18] A. Kollross, A classification of hyperpolar and cohomogeneity one actions, *Trans. Amer. Math. Soc.* **354** (2002), 571–612.
- [19] T. Levi-Civita, Famiglie di superficie isoparametriche nell'ordinario spazio euclideo, *Rend. Acc. Naz. Lincei* **XXVI** (1937), 355–362.
- [20] G.D. Mostow, Some new decomposition theorems for semi-simple groups, *Mem. Amer. Math. Soc.* **14** (1955), 31–54.
- [21] G.D. Mostow, On maximal subgroups of real Lie groups, *Ann. of Math.* **74** (1961), 503–517.
- [22] A.L. Onishchik and E.B. Vinberg (Eds.), *Lie groups and Lie algebras III*, Encyclopaedia of Mathematical Sciences Vol. 41, Springer-Verlag, Berlin, Heidelberg, 1994.
- [23] B. Segre, Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni, *Rend. Acc. Naz. Lincei* **XXVII** (1938), 203–207.
- [24] C. Somigliana, Sulle relazione fra il principio di Huygens e l'ottica geometrica, *Atti. Acc. Sc. Torino* **LIV** (1918-1919), 974–979.
- [25] R. Takagi, On homogeneous real hypersurfaces in a complex projective space, *Osaka J. Math.* **10** (1973), 495–506.
- [26] H. Tamaru, The local orbit types of symmetric spaces under the actions of the isotropy subgroups, *Differential Geom. Appl.* **11** (1999), 29–38.
- [27] H.M. Tao, The maximal nonsemisimple subalgebras of a noncompact real semisimple Lie algebra (in Chinese), *Acta Math. Sinica* **16** (1966), 253–268.
- [28] K. Tsukada, Totally geodesic hypersurfaces of naturally reductive homogeneous spaces, *Osaka J. Math.* **33** (1996), 697–707.

DEPARTMENT OF MATHEMATICS, KING'S COLLEGE LONDON, STRAND, LONDON, WC2R 2LS, UNITED KINGDOM

E-mail address: jurgen.berndt@kcl.ac.uk

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, 1-3-1 KAGAMIYAMA, HIGASHI-HIROSHIMA, 739-8526, JAPAN

E-mail address: tamaru@math.sci.hiroshima-u.ac.jp